# Fuzzy finite element method for vibration analysis of imprecisely defined bar

**A THESIS**

**Submitted in partial fulfillment of the requirements for the award of the degree of**

**MASTER OF SCIENCE**

**In**

**MATHEMATICS**

**By**

**Nisha Rani Mahato**

**Under the supervision of**

**Prof. S. Chakraverty**

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**DEPARTMENT OF MATHEMATICS**

**NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA**

**ROURKELA-769008**

**ODISHA, INDIA**

**NATIONAL INSTITUTE OF TECHNOLOGY**

**ROURKELA**

## DECLARATION

I hereby certify that the work which is being presented in the thesis entitled “Fuzzy finite element method for vibration analysis of imprecisely defined bar” in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Dr. S. Chakraverty.

The matter embodied in this has not been submitted by me for the award of any other degree.

Date: (NISHA RANI MAHATO)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Dr. S. CHAKRAVERTY

Professor, Department of Mathematics

National Institute of Technology

Rourkela – 769008

Odisha, India

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And to all mighty, who made all things possible………

(NISHA RANI MAHATO)

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# Abstract

This thesis investigates the vibration of a bar for computing its natural frequency with interval or fuzzy material properties in the finite element method. The problem is formulated first using the energy equation by converting the problem to a generalized eigenvalue problem. The generalized eigenvalue problem obtained contains the mass and stiffness matrix. In general these matrices contain the crisp values of the parameters and then it is easy to solve by various well known methods. But, in actual practice there are incomplete information about the variables being a result of errors in measurements, observations, applying different operating conditions or it may be maintenance induced error, etc. Rather than the particular value of the material properties we may have only the bounds of the values. These bounds may be given in term of interval or fuzzy. Thus we will have the finite element equations having the interval and fuzzy stiffness and mass matrices. So, in turn one has to solve by interval or fuzzy generalized eigenvalue problem. As such detail study related to interval and fuzzy computation has been done. First crisp values of material properties are considered. Then the problem has been undertaken taking the properties as interval and fuzzy. Initially, Young's modulus and density as material properties have been considered as interval in two different cases, one for homogenous and other one for non-homogenous material properties. Then the problem has been analyzed using Young's modulus and density properties as fuzzy. First Fuzzy material properties in terms of fuzzy number that is triangular fuzzy number is considered then trapezoidal fuzzy number is considered in the finite element method. The fuzzy material properties are solved using $\alpha$-cut to obtain the corresponding intervals. Then using interval computation natural frequencies are obtained and the fuzzy results are depicted in term of plots.

# 1 Introduction

Finite Element Method is being extensively used to find approximate results of complicated structures of which exact solutions cannot be found. The finite element method for the vibration problem is a method of finding approximate solutions of the governing partial differential equations by transforming it into an eigenvalue problem.

For various scientific and engineering problems, it is an important issue how to deal with variables and parameters of uncertain value. Generally, the parameters are taken as constant for simplifying the problem. But instead of the particular value of the material properties we have only the bounds of the values due to vagueness.

## 1.1 Finite element model

The finite element method is a numerical method for finding approximate solutions of partial differential equations. The solution approach is based either on elimination of the differential equation completely or rendering the PDE into an approximating system of ordinary differential equations which are then numerically integrated using standard techniques. Finite Element method can be applied to structures, biomechanics and electromagnetic field problems. Simple linear static problems and highly complex nonlinear transient dynamic problems are effectively solved using the finite element method.

FEM helps in detailed visualization of bend or twist in the structures, and indicates the distribution of displacements. **Fig.1** shows the finite element model of composite inboard.

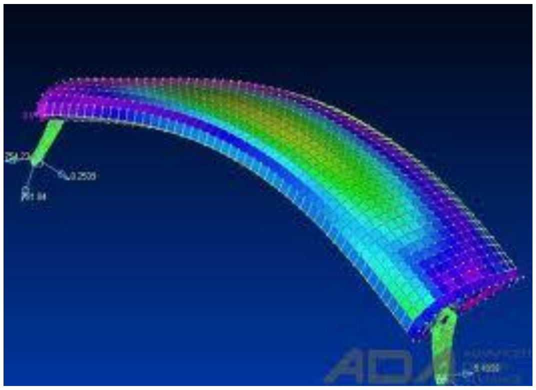


**Fig.1** Finite element model of composite inboard

## 1.2 Interval, fuzzy set and fuzzy numbers

**Interval:** An interval is a subset of $R$ such that $A = [a_1, a_2] = \{t \mid a_1 \le t \le a_2, a_1, a_2 \in R\}$.

If $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are two intervals, then the arithmetic operations are:

- $A + B = [a_1 + b_1, a_2 + b_2]$
- $A - B = [a_1 - b_2, a_2 - b_1]$
- $A \bullet B = [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$
- $A / B = [a_1, a_2][1/b_1, 1/b_2]$

**Fuzzy set:** A fuzzy set can be defined as the set of ordered pairs such that $A = \{(x, \mu_A(x)) / x \in X, \mu_A(x) \in [0,1]\}$, where $\mu_A(x)$ is called the membership function of $x$.

**Fuzzy Number:** A fuzzy number is a convex normalized fuzzy set of the crisp set such that for only one $x \in X$, $\mu_A(x) = 1$ and $\mu_A(x)$ is piecewise continuous.

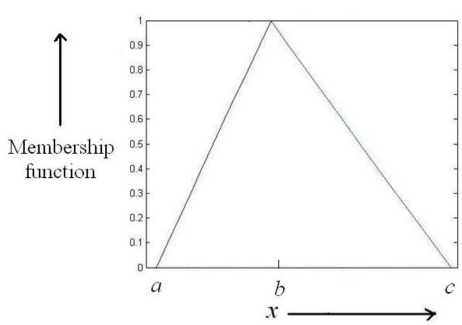



**Fig .2** Triangular fuzzy number

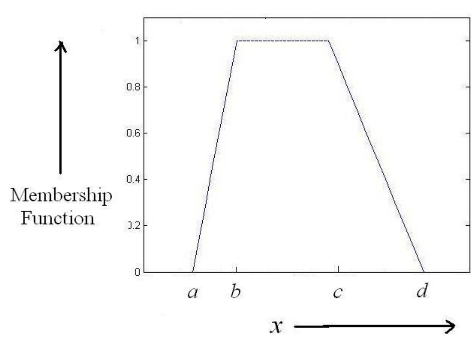



**Fig.3** Trapezoidal fuzzy number

**Triangular Fuzzy Number:** A triangular fuzzy number (TFN) given by $A=(a,b,c)$ as shown in **Fig.2** is a special case of fuzzy number and its membership function is given by $\mu_A(x)$ where $\mu_A(x)\in[0,1]$ such that

$$\mu_A(x)=\begin{cases} 0 & ,x<a \\ \dfrac{x-a}{b-a} & ,x\in[a,b] \\ \dfrac{c-x}{c-b} & ,x\in[b,c] \\ 0 & ,x>c \end{cases}$$

The interval $A=[a,c]$ can be obtained simply by substituting $\alpha=0$ and the crisp result can be obtained by substituting $\alpha=1$ in fuzzy case.

Its interval form in given as $A=[a+\alpha(b-a),c-\alpha(c-b)]$ where $\alpha\in[0,1]$.

**Trapezoidal Fuzzy Number:** A trapezoidal fuzzy number (TrFN) given by $A=(a,b,c,d)$ as shown in **Fig.3** is a fuzzy number whose membership function is given by $\mu_A(x)$ such that

$$\mu_A(x)=\begin{cases} 0 & ,x<a \\ \dfrac{x-a}{b-a} & ,x\in[a,b] \\ 1 & ,x\in[b,c] \\ \dfrac{c-x}{c-b} & ,x\in[c,d] \\ 0 & ,x>d \end{cases}$$

The interval $A=[a,d]$ can be obtained simply by substituting $\alpha=0$ in fuzzy case and for $b=c$ we can get the triangular fuzzy number.

Its interval form in given as $A=[a+\alpha(b-a),d-\alpha(d-c)]$ where $\alpha\in[0,1]$.

## 2 Literature Review

Recently investigations are carried out by various researchers throughout the globe by using the uncertainty or the fuzziness of the material properties.

Various generalized model of uncertainty have been applied to finite element analysis to solve the vibration and static problems by using interval or fuzzy parameters. Although FEM in vibration problem is well known and there exist large number of papers related to this. As such few papers that are related to Fuzzy FEM are discussed here. Elishakoff et al. [1] investigated the turning around a method of successive iterations to yield closed-form solutions for vibrating inhomogeneous bars, where the author studied the method of successive approximations so as to obtain closed form solutions for vibrating inhomogeneous bars. Panigrahi et al. [2] presented a discussion about the vibration based damage detection in a uniform strength beam using genetic algorithm. Dimarogonas [3] studied the interval analysis of vibrating systems, where the author presented the theory for vibrating system taking interval rotator dynamics. Naidoo [4] studied the application of intelligent technology on a multi-variable dynamical system, where a fuzzy logic control algorithm was implemented to test the performance in temperature control. The generalized fuzzy eigen value problem is investigated by Chiao [5] who used the extension principle to find out the solution. Fuzzy finite element analysis for imprecisely defined system is presented by Rao and Sawyer [6]. Recently Gersem et al. [7] presented a discussion about the non-probabilistic fuzzy finite element method for the dynamic behavior of structures using uncertain parameter. Verhaeghe et al. [8] discussed the static analysis of structures using fuzzy finite analysis technique based on interval field. Very recently Mahato et al. [9] studied the properties of fixed free bar with Fuzzy Finite Element Method for computing its natural frequency.

# 3 Aim

In this project a bar has been considered to describe the finite element method and then the interval Finite element method followed by fuzzy finite element method is discussed for the problem. As already mentioned, generally the values of variables or properties are taken as crisp but in actual case the accurate crisp values cannot be obtained. To overcome the vagueness we use interval and fuzzy numbers in place of crisp values. Simulation with various numbers of elements with crisp, interval and fuzzy material properties in the vibration of a bar has been investigated here. As such corresponding generalized eigenvalue problem with interval and fuzzy numbers has been solved to investigate the problem.

# 4 Structural finite element model for a bar

The structure is considered to be an one-dimensional fixed free bar as shown in **Fig. 4** with the governing equation

$$AE\frac{\partial^2 U}{\partial x^2} = \rho A\frac{\partial^2 U}{\partial t^2} \tag{1}$$

Here $E$, $A$, $\rho$ and $U$ are Young's modulus of elasticity, cross sectional area, density and displacement at any point respectively.

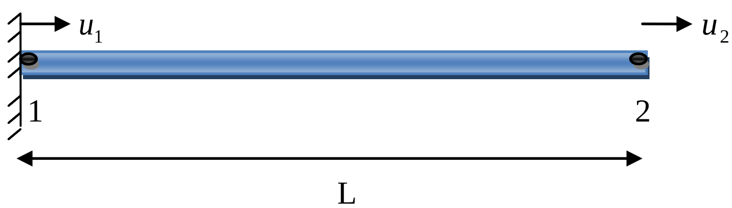



**Fig.4** A fixed free bar element

We develop the necessary equation for single element (Zienkiewicz [10], Klaus [11], Seshu [12]) using shape functions- $\phi_1(x) = 1 - \frac{x}{l}$ and $\phi_2(x) = \frac{x}{l}$.

The displacement at any point is given by $U(x,t) = \left(1 - \frac{x}{l}\right)u_1(t) + \frac{x}{l}u_2(t)$

Accordingly the kinetic energy $T$ and potential energy $V$ are given respectively by

$$T = \frac{1}{2}\int_0^l \rho A dx \left(\dot{U}(x,t)\right)^2 = \frac{1}{6}ml\left(\dot{u}_1^2(t) + \dot{u}_2^2(t) + \dot{u}_1(t)\dot{u}_2(t)\right) \tag{2}$$

$$V = \frac{1}{2}\int_0^l EA\left(\frac{\partial U}{\partial x}\right)^2 dx = \frac{EA}{2l}\left(u_1^2(t) + u_2^2(t) - 2u_1(t)u_2(t)\right) \tag{3}$$

Using Lagrange's Equation $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$ with Lagrange's operator $L=T-V$ and generalized coordinator, $q=u_1(t),u_2(t)$, we obtain the equation of motion as

$$[M]\{\ddot{U}(t)\}+[K]\{U(t)\}=\{0\} \tag{4}$$

In matrix form (Zienkiewicz [10], Klaus [11], Seshu [12]), it can be written as

$$\frac{\rho Al}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}+\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For free vibration, taking $U = Ae^{iwt}$, where $A$ is the vector of the nodal displacements, we have

$$[K]\{A\}=\omega^2[M]\{A\} \tag{5}$$

This is a typical eigenvalue problem and is solved numerically as a generalized eigenvalue problem.

## 5 Finite element model for homogeneous bar with crisp material properties

A fixed free bar having crisp values of material properties is considered for determining the natural frequency. The bar is simulated numerically with finite element models taking one, two, three and four element and for each element mass and stiffness matrix is written satisfying the boundary condition $u_1 = 0$. Then natural frequencies are obtained after getting the global mass and stiffness matrices through assembling. The eigenvalue equations for various elements according to the boundary condition may easily be written. As such for one, two, three and four element equations are given here in Eqs. (6) to (9) respectively.

$$2\frac{EA}{L}u_2 = \omega^2 \frac{\rho AL}{6}u_2 \quad \text{(P. Seshu [12])} \tag{6}$$

$$2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \omega^2 \frac{\rho AL}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{7}$$

$$3\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \omega^2 \frac{\rho AL}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{8}$$

$$4\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \omega^2 \frac{\rho AL}{24}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \tag{9}$$

Taking the values of the parameters as $E = 2\times10^{11}\, N/m^2$, $\rho = 7800\, kg/m^3$, $A = 30\times10^{-6}\, m^2$ and $L = 1m$, natural frequencies are obtained from Eqs. (6) to (9) and are given in **Table 1**

**Table 1** Crisp value of frequencies with $E, \rho, A, L$ as crisp

| | Number of elements | | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Modes | 1 | 8770.6 | 8160.724 | 8045 | 7791 |
| | 2 | | 28505 | 26312 | 25596 |
| | 3 | | | 47733 | 45059 |
| | 4 | | | | 71475 |

In the subsequent sections imprecisely defined bar viz. taking the material properties either in term of interval or fuzzy for homogenous and non-homogenous cases are discussed.

## 6 Interval Finite element model for homogeneous bar

Here interval values of the material properties are considered.

From Eq.(5) we get the eigenvalue problem for interval values as

$$\left[\underline{K},\overline{K}\right]\{A\}=[\underline{\omega},\overline{\omega}]^2\left[\underline{M},\overline{M}\right]\{A\} \tag{10}$$

which reduces to $\left[\underline{K}\right]\{A\}=\underline{\omega}_1{}^2\left[\overline{M}\right]\{A\}$ and $\left[\overline{K}\right]\{A\}=\overline{\omega}_1{}^2\left[\underline{M}\right]\{A\}$ (11)

$$\left[\underline{K}\right]\{A\}=\overline{\omega}_2{}^2\left[\underline{M}\right]\{A\} \text{and } \left[\overline{K}\right]\{A\}=[\underline{\omega}_2]^2\left[\overline{M}\right]\{A\} \tag{12}$$

Here $\underline{\omega}_1<\underline{\omega}_2<\overline{\omega}_2<\overline{\omega}_1$. Now taking $\underline{\omega}=\max(\underline{\omega}_1,\underline{\omega}_2)$ and $\overline{\omega}=\min(\overline{\omega}_1,\overline{\omega}_2)$ we get the interval as $[\underline{\omega},\overline{\omega}]$. First using the eigenvalue Eq.(11) the natural frequencies are obtained for different material properties as intervals. Then using Eqs. (11) and (12) the natural frequencies are obtained only for both Young's modulus and density as intervals.

### 6.1 Homogenous fixed free bar with Young's modulus as an interval

Taking $E=\left[\underline{E},\overline{E}\right]$, the governing equations for one, two, three and four elements according to the same boundary condition are computed. One, two, three and four element equations are incorporated here in Eqs. (13) to (16) respectively,

$$\frac{\underline{E}A}{L}=\underline{\omega}^2\frac{\rho AL}{3} \text{and } \frac{\overline{E}A}{L}=\overline{\omega}^2\frac{\rho AL}{3} \tag{13}$$

$$2\frac{A}{L}\begin{bmatrix}2\underline{E} & -\overline{E}\\ -\overline{E} & \underline{E}\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\underline{\omega}^2\frac{\rho AL}{12}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} \text{and } 2\frac{A}{L}\begin{bmatrix}2\overline{E} & -\underline{E}\\ -\underline{E} & \overline{E}\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\overline{\omega}^2\frac{\rho AL}{12}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} \tag{14}$$

$$\frac{3A}{L}\begin{bmatrix}2\underline{E} & -\overline{E} & 0\\ -\overline{E} & 2\underline{E} & -\overline{E}\\ 0 & -\overline{E} & \underline{E}\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix}=\underline{\omega}^2\frac{\rho AL}{18}\begin{bmatrix}4 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix} \text{ and}$$

$$\frac{3A}{L}\begin{bmatrix} 2\overline{E} & -\underline{E} & 0 \\ -\underline{E} & 2\overline{E} & -\underline{E} \\ 0 & -1 & \overline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \overline{\omega}^2\,\frac{\rho AL}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{15}$$

$$\frac{4A}{L}\begin{bmatrix} 2\underline{E} & -\overline{E} & 0 & 0 \\ -\overline{E} & 2\underline{E} & -\overline{E} & 0 \\ 0 & -\overline{E} & 2\underline{E} & -\overline{E} \\ 0 & 0 & -\overline{E} & \underline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \underline{\omega}^2\,\frac{\rho AL}{24}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \text{ and}$$

$$\frac{4A}{L}\begin{bmatrix} 2\overline{E} & -\underline{E} & 0 & 0 \\ -\underline{E} & 2\overline{E} & -\underline{E} & 0 \\ 0 & -\underline{E} & 2\overline{E} & -\underline{E} \\ 0 & 0 & -\underline{E} & \overline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \overline{\omega}^2\,\frac{\rho AL}{24}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \tag{16}$$

For values $L = 1m,\ E = \left[\underline{E}, \overline{E}\right] = \left[1.998\times10^{11}, 2.002\times10^{11}\right] N/m^2$, $\rho = 7800\ kg/m^3$ and $A = 30\times10^{-6}\, m^2$, using eigenvalue Eq.(11) the natural frequencies are obtained from Eqs. (13) to (16) and are given in **Table 2**.

**Table 2** Interval values of frequencies with Young's modulus as interval

| | | Number of elements | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Modes | 1 | ( 8766.2, 8775 ) | ( 8136, 8183 ) | ( 7989, 8101 ) | ( 7692, 7889 ) |
| | 2 | | ( 28503, 28508 ) | (26299, 26325 ) | ( 25567, 25624 ) |
| | 3 | | | ( 47133, 48679 ) | ( 45049, 45069 ) |
| | 4 | | | | ( 71473, 71476 ) |

## 6.2 Homogenous fixed free bar with density as an interval

A homogenous fixed free bar with density as interval is considered now. The governing eigenvalue equations satisfying the boundary condition for various elements are obtained. One, two, three and four element equations are incorporated here in Eqs. (17) to (20) respectively,

$$\frac{EA}{L}u_2 = \underline{\omega}^2 \frac{\overline{\rho}AL}{3} \text{ and } \frac{EA}{L}u_2 = \overline{\omega}^2 \frac{\underline{\rho}AL}{3} \tag{17}$$

$$\frac{EA}{L}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \underline{\omega}^2 \frac{AL}{12}\begin{bmatrix} 4\overline{\rho} & \overline{\rho} \\ \overline{\rho} & 2\overline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \text{and}$$

$$\frac{EA}{L}\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \overline{\omega}^2 \frac{AL}{12}\begin{bmatrix} 4\underline{\rho} & \underline{\rho} \\ \underline{\rho} & 2\underline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{18}$$

$$\frac{3AE}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underline{\omega}^2 \frac{AL}{18}\begin{bmatrix} 4\overline{\rho} & \overline{\rho} & 0 \\ \overline{\rho} & 4\overline{\rho} & \overline{\rho} \\ 0 & \overline{\rho} & 2\overline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \text{and}$$

$$\frac{3AE}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \overline{\omega}^2 \frac{AL}{18}\begin{bmatrix} 4\underline{\rho} & \underline{\rho} & 0 \\ \underline{\rho} & 4\underline{\rho} & \underline{\rho} \\ 0 & \underline{\rho} & 2\underline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{19}$$

$$\frac{4EA}{L}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \underline{\omega}^2 \frac{AL}{24}\begin{bmatrix} 4\overline{\rho} & \overline{\rho} & 0 & 0 \\ \overline{\rho} & 4\overline{\rho} & \overline{\rho} & 0 \\ 0 & \overline{\rho} & 4\overline{\rho} & \overline{\rho} \\ 0 & 0 & \overline{\rho} & 2\overline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \text{ and}$$

$$\frac{4EA}{L}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \overline{\omega}^2 \frac{AL}{24}\begin{bmatrix} 4\underline{\rho} & \underline{\rho} & 0 & 0 \\ \underline{\rho} & 4\underline{\rho} & \underline{\rho} & 0 \\ 0 & \underline{\rho} & 4\underline{\rho} & \underline{\rho} \\ 0 & 0 & \underline{\rho} & 2\underline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \tag{20}$$

For values $L = 1m, E = 2\times 10^{11} N/m^2, \rho = [\underline{\rho}, \overline{\rho}] = [7500,8000]\, kg/m^3$ and $A = 30\times 10^{-6}\, m^2$, the computed natural frequencies are given in **Table 3** for various element equations.

**Table 3** Interval values of frequencies with density as interval

| | | Number of elements | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Modes | 1 | ( 8660.3, 8944.3 ) | ( 8057, 8321 ) | ( 7944, 8205 ) | ( 7693, 7946) |
| | 2 | | ( 28147, 29070 ) | (25981, 26833 ) | ( 25274, 26103 ) |
| | 3 | | | ( 47133, 48679 ) | ( 44493, 45952 ) |
| | 4 | | | | ( 70575, 72890 ) |

## 6.3 Homogenous bar with density ($\rho$) and Young's modulus ($E$) as intervals

In this case, the same bar with both density and Young's modulus as interval (Jaulin et al. [13]) is considered. The governing equations satisfying the boundary condition are obtained again where $\rho$ and $E$ are considered as interval i.e. $\rho=[\underline{\rho},\overline{\rho}]$ and $E=[\underline{E},\overline{E}]$. One, two, three and four element equations are incorporated here in Eqs. (21) to (24) respectively,

$$\frac{\underline{E}A}{L}u_2=\underline{\omega}^2\frac{\overline{\rho}AL}{3} \text{ and } \frac{EA}{L}u_2=\overline{\omega}^2\frac{\underline{\rho}AL}{3} \tag{21}$$

$$\frac{A}{L}\begin{bmatrix} 4\underline{E} & -2\overline{E} \\ -2\overline{E} & 2\underline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \underline{\omega}^2 \frac{AL}{12}\begin{bmatrix} 4\overline{\rho} & \overline{\rho} \\ \overline{\rho} & 2\overline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \text{and}$$

$$\frac{A}{L}\begin{bmatrix} 4\overline{E} & -2\underline{E} \\ -2\underline{E} & 2\overline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \overline{\omega}^2 \frac{AL}{12}\begin{bmatrix} 4\underline{\rho} & \underline{\rho} \\ \underline{\rho} & 2\underline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{22}$$

$$\frac{3A}{L}\begin{bmatrix} 2\underline{E} & -\overline{E} & 0 \\ -\overline{E} & 2\underline{E} & -\overline{E} \\ 0 & -\overline{E} & \underline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underline{\omega}^2 \frac{AL}{18}\begin{bmatrix} 4\overline{\rho} & \overline{\rho} & 0 \\ \overline{\rho} & 4\overline{\rho} & \overline{\rho} \\ 0 & \overline{\rho} & 2\overline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \text{and}$$

$$\frac{3A}{L}\begin{bmatrix} 2\overline{E} & -\underline{E} & 0 \\ -\underline{E} & 2\overline{E} & -\underline{E} \\ 0 & -1 & \overline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \overline{\omega}^2 \frac{AL}{18}\begin{bmatrix} 4\underline{\rho} & \underline{\rho} & 0 \\ \underline{\rho} & 4\underline{\rho} & \underline{\rho} \\ 0 & \underline{\rho} & 2\underline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{23}$$

$$\frac{4A}{L}\begin{bmatrix} 2\underline{E} & -\overline{E} & 0 & 0 \\ -\overline{E} & 2\underline{E} & -\overline{E} & 0 \\ 0 & -\overline{E} & 2\underline{E} & -\overline{E} \\ 0 & 0 & -\overline{E} & \underline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \underline{\omega}^2 \frac{AL}{24}\begin{bmatrix} 4\overline{\rho} & \overline{\rho} & 0 & 0 \\ 1 & 4\overline{\rho} & \overline{\rho} & 0 \\ 0 & \overline{\rho} & \overline{\rho} & \overline{\rho} \\ 0 & 0 & \overline{\rho} & 2\overline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \text{ and}$$

$$\frac{4A}{L}\begin{bmatrix} 2\overline{E} & -\underline{E} & 0 & 0 \\ -\underline{E} & 2\overline{E} & -\underline{E} & 0 \\ 0 & -\underline{E} & 2\overline{E} & -\underline{E} \\ 0 & 0 & -\underline{E} & \overline{E} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \overline{\omega}^2 \frac{AL}{24}\begin{bmatrix} 4\underline{\rho} & \underline{\rho} & 0 & 0 \\ \underline{\rho} & 4\underline{\rho} & \underline{\rho} & 0 \\ 0 & \underline{\rho} & 4\underline{\rho} & \underline{\rho} \\ 0 & 0 & \underline{\rho} & 2\underline{\rho} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \tag{24}$$

Taking values $L = 1m, E = [\underline{E}, \overline{E}] = [1.998\times10^{11}, 2.002\times10^{11}] N/m^2$, $A = 30\times10^{-6}\, m^2$ and $\rho = [\underline{\rho}, \overline{\rho}] = [7500, 8000]\, kg/m^3$.

According to Eq. (11), the natural frequencies are computed for various elements and are incorporated in **Table 4.1**.

**Table 4.1** Interval values of frequencies with $\rho, E$ as intervals

| | | Number of elements | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Modes | 1 | ( 8655.9, 8948.7 ) | ( 8034, 8346 ) | (7888, 8261 ) | (7596, 8046) |
| | 2 | | ( 28144, 29072 ) | (25968, 26846 ) | ( 25245, 26132 ) |
| | 3 | | | ( 47131, 48680 ) | ( 44482, 45962 ) |
| | 4 | | | | ( 70574, 72891 ) |

Next the second method viz. using Eqs. (11) and (12), the natural frequencies are computed for various elements and are incorporated in **Table 4.2**.

**Table 4.2** Interval values of frequencies with $\rho, E$ as intervals

| | Number of elements | | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Modes | 1 | ( 8664.6, 8939.8 ) | ( 8081, 8297 ) | (7999, 8147 ) | (7790, 7845 |
| | 2 | | ( 28149, 29067 ) | (25994, 26819 ) | ( 25302, 26073 ) |
| | 3 | | | ( 47135, 48677 ) | ( 44503, 45941 ) |
| | 4 | | | | ( 70577, 72890 ) |

## 7 Fuzzy Finite element model of Homogenous bar for triangular fuzzy number

In this head fuzzy values (in term of Triangular fuzzy number) of the material properties are considered.

### 7.1 Homogenous fixed free bar with fuzzy Young's modulus

A homogenous fixed free bar with fuzzy value of Young's modulus is considered here. If $E=(a_1,b_1,c_1)$ is a triangular fuzzy number then it can be written in interval form as $\left[\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)\right]$, where $\alpha\in[0,1]$. The governing equations satisfying the boundary condition $u_1=0$ are same as in interval case in Eqs. (13) to (16) with $E=\left[\underline{E},\overline{E}\right]=\left[\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)\right]$.

Here the values of the parameters are considered here as:

$L=1m, E=(1.998\times10^{11}, 2\times10^{11}, 2.002\times10^{11})\ N/m^2, \rho=7800\ kg/m^3$ and $A=30\times10^{-6}m^2$.

Corresponding natural frequencies are computed for one, two, three and four element which are fuzzy numbers (TFN). **Figs. 5** to **8** depict the fuzzy natural frequency plot for different elements.

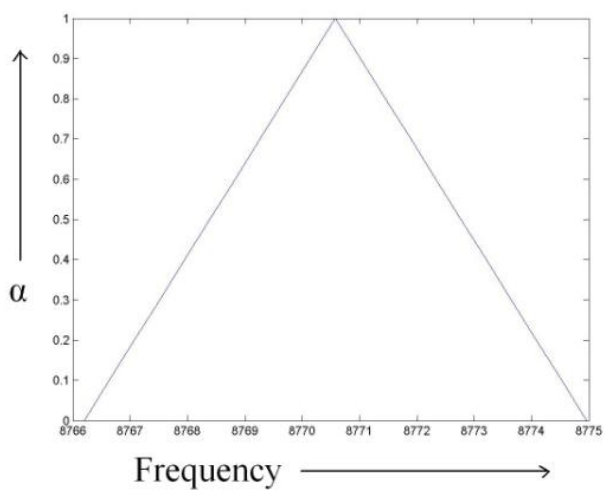



**Fig. 5** Natural Frequency (one element)

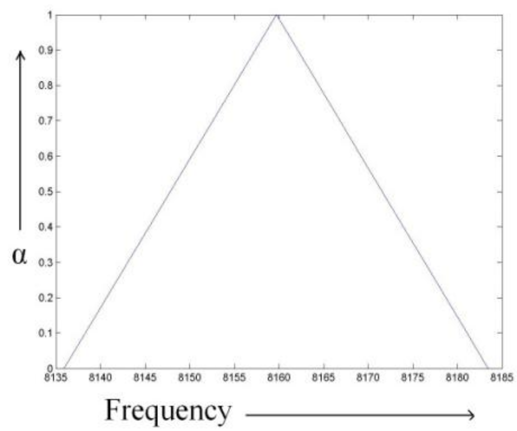



**Fig. 6(a)** First frequency (two element)

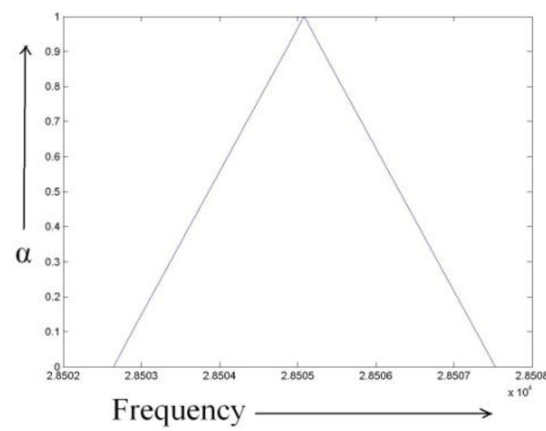



**Fig. 6(b)** Second frequency (two element)

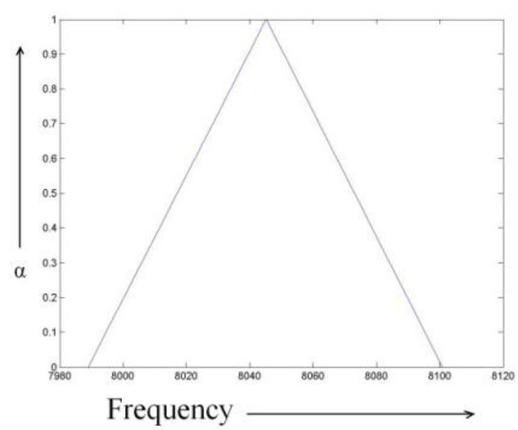



**Fig. 7(a)** First frequency (three element)

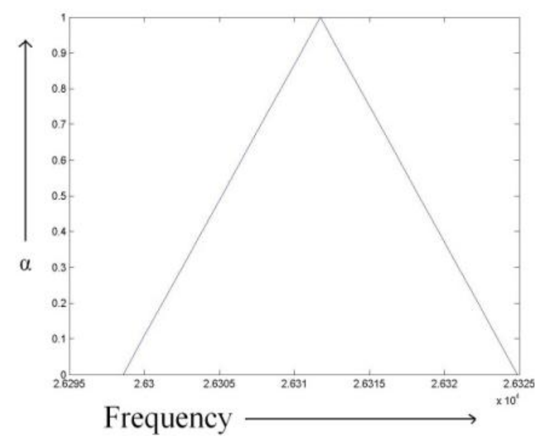



**Fig. 7(b)** Second frequency (three element)

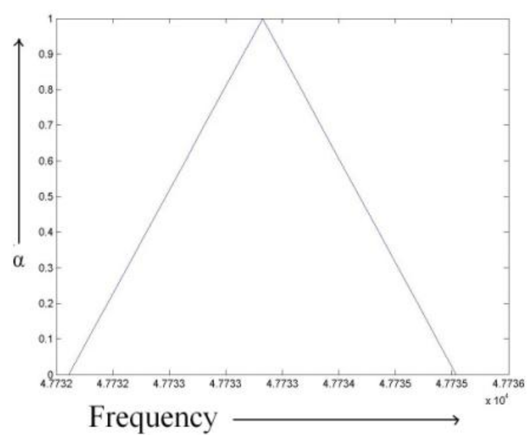



**Fig. 7(c)** Third frequency (three element)

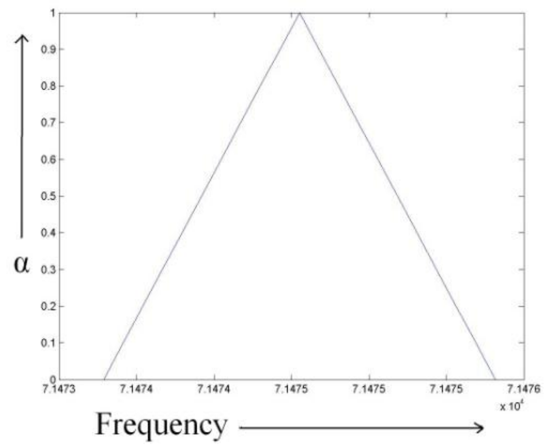



**Fig. 8(a)** First frequency (four element)

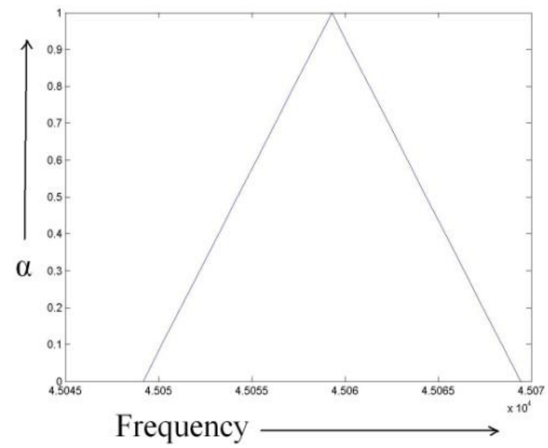



**Fig. 8(b)** Second frequency (four element)

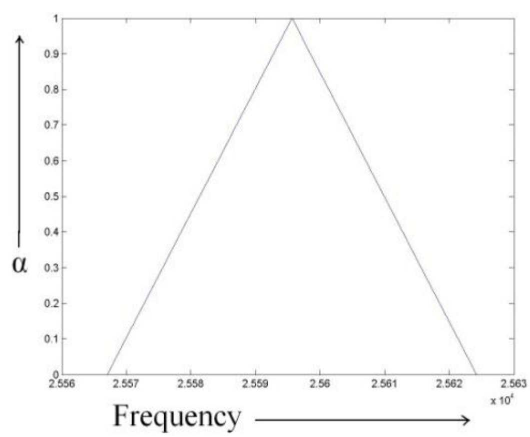



**Fig. 8(c)** Third frequency (four element)

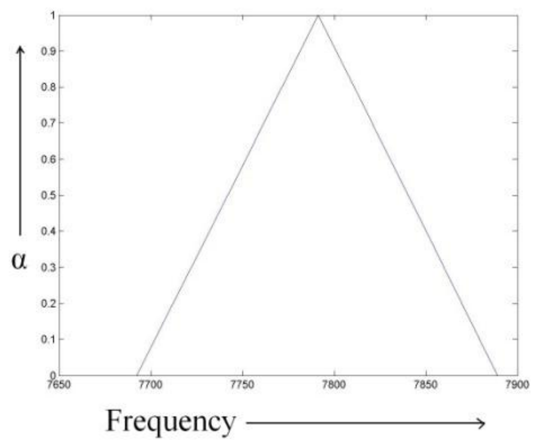



**Fig. 8(d)** Fourth frequency (four element)

## 7.2 Homogenous fixed free bar with fuzzy density

Here the same homogenous bar with fuzzy value of density is considered. Accordingly if $\rho = (a_2, b_2, c_2)$ is a triangular fuzzy number then its interval form is $[\alpha(b_2 - a_2) + a_2, c_2 - \alpha(c_2 - b_2)]$. The governing equations satisfying the boundary condition are same as in interval case in Eqs. (17) to (20) with $\rho = [\underline{\rho}, \overline{\rho}] = [\alpha(b_2 - a_2) + a_2, c_2 - \alpha(c_2 - b_2)]$.

Values of the parameters are considered in this case as:

$L = 1m$, $E = 2 \times 10^{11}\, N/m^2$, $\rho = (7500, 7800, 8000)\, kg/m^3$ and $A = 30 \times 10^{-6}\, m^2$. Corresponding natural frequencies are computed for one, two, three and four element which are fuzzy numbers. The fuzzy natural frequency plots for different elements are depicted in term of plots from **Figs. 9** to **12**.

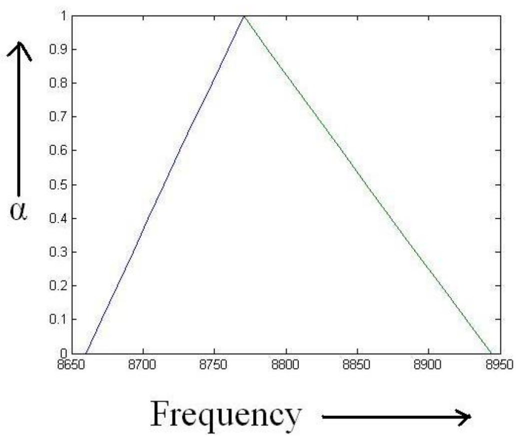



**Fig. 9** Natural frequency (one element)

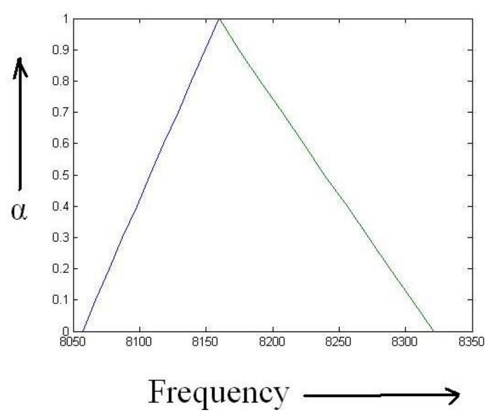



**Fig. 10(a)** First frequency (two element)

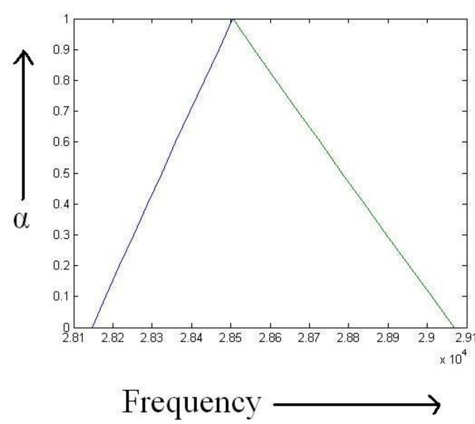



**Fig. 10(b)** Second frequency (two element)

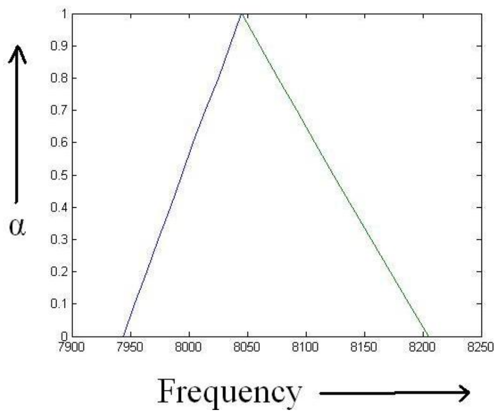



**Fig. 11(a)** First frequency (three element)

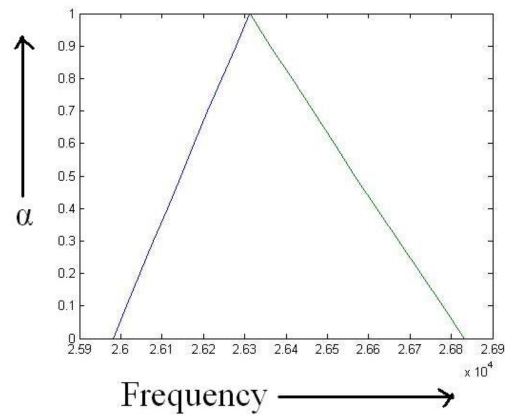



**Fig. 11(b)** Second frequency (three element)

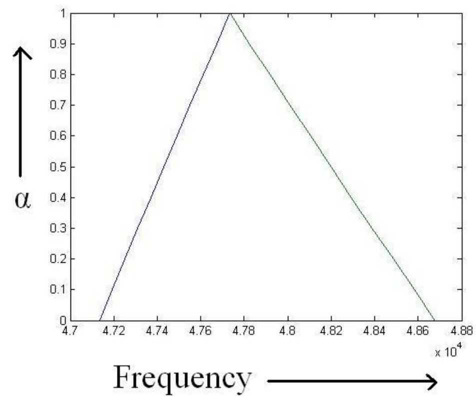



**Fig. 11(c)** Third frequency (three element)

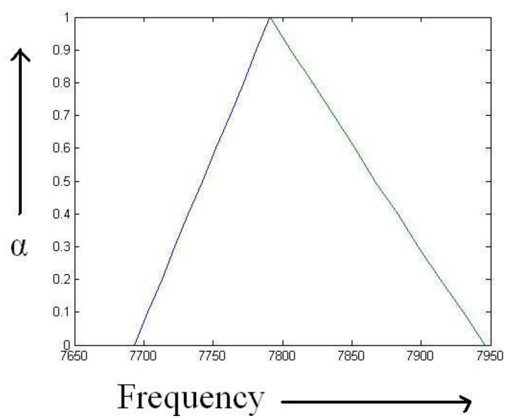



**Fig. 12(a)** First frequency (four element)

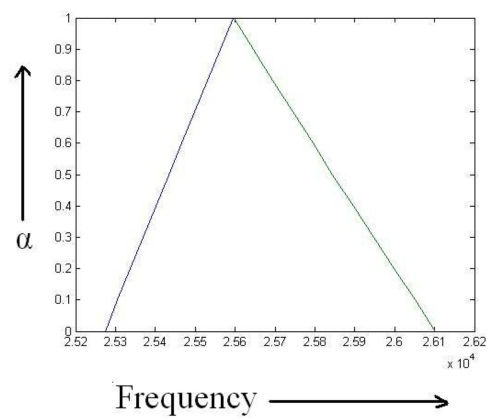



**Fig. 12(b)** Second frequency (four element)

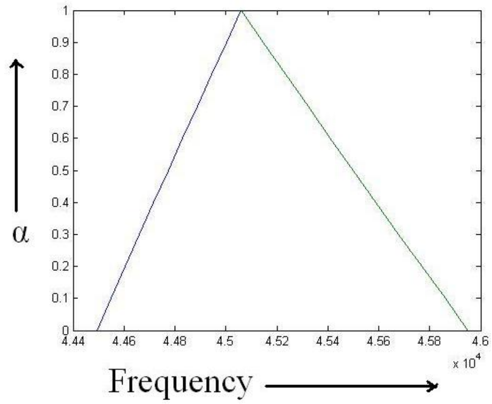



**Fig. 12(c)** Third frequency (four element)

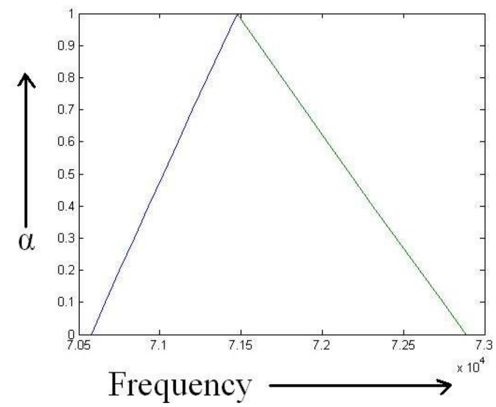



**Fig. 12(d)** Fourth frequency (four element)

### 7.3 Homogenous fixed free bar with fuzzy Young's modulus and fuzzy density

A fixed free bar with fuzzy values of density and Young's modulus is considered. If $E=(a_1,b_1,c_1)$ and $\rho=(a_2,b_2,c_2)$ are Triangular Fuzzy Numbers (TFN) (Ross [14]) then their corresponding interval forms in term of α-cut are $[\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)]$ and $[\alpha(b_2-a_2)+a_2, c_2-\alpha(c_2-b_2)]$. The eigenvalue equations satisfying the boundary condition are same as in interval case in Eqs. (21) to (24) with $\underline{E}=\alpha(b_1-a_1)+a_1$, $\overline{E}=c_1-\alpha(c_1-b_1)$, $\underline{\rho}=\alpha(b_2-a_2)+a_2$ and $\overline{\rho}=c_2-\alpha(c_2-b_2)$.

The values of the parameters are considered as: $\rho=(7500,7800,8000)\ kg/m^3$, $L=1m$,
$E=(1.998\times10^{11}, 2\times10^{11}, 2.002\times10^{11})\ N/m^2$, and $A=30\times10^{-6}\ m^2$.

According to Eq.(11) corresponding natural frequencies are computed which are triangular fuzzy number (TFN) and depicted by **Figs. 13** to **16**

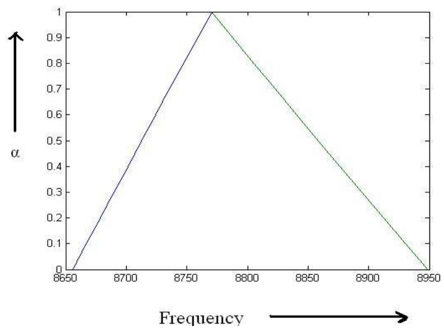



**Fig. 13** First frequency (one element)

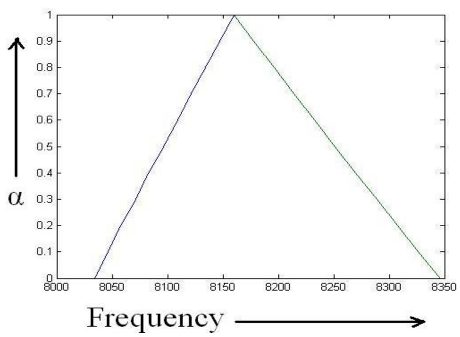



**Fig. 14(a)** First frequency (two element)

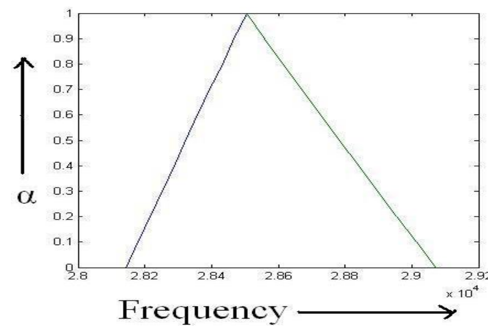



**Fig. 14(b)** Second frequency (two element)

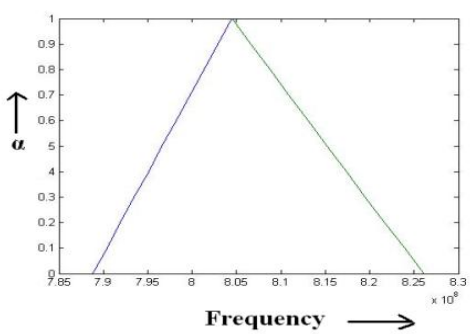



**Fig. 15(a)** First frequency (Three element)

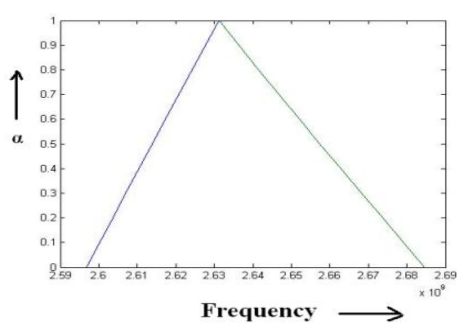



**Fig. 15(b)** Second frequency (Three element)

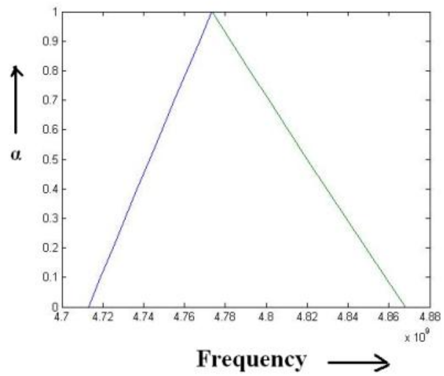



**Fig. 15(c)** Third frequency (Three element)

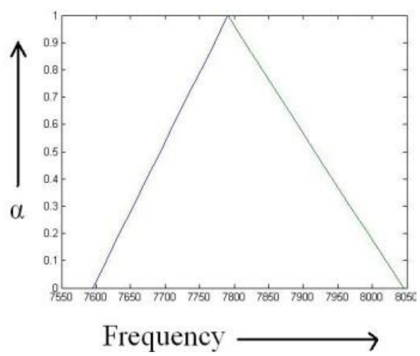



**Fig. 16(a)** First frequency (Four element)

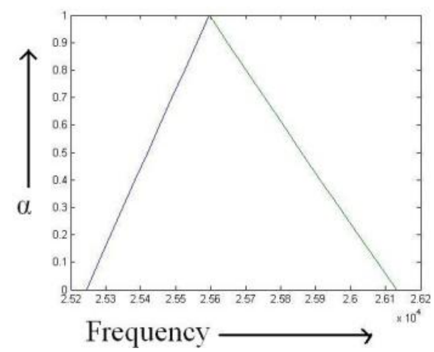



**Fig. 16(b)** Second frequency (Four element)

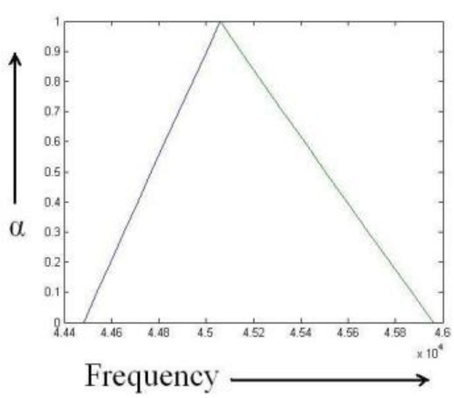



**Fig. 16(c)** Third frequency (Four element)

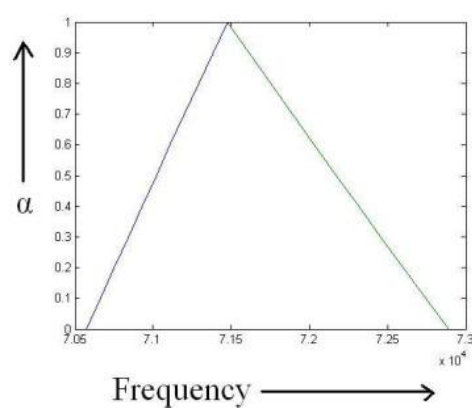



**Fig. 16(d)** Fourth frequency (Four element)

The eigenvalue equations according to Eqs. (11) and (12) satisfying the boundary condition are computed as in interval case in Eqs. (21) to (24) to obtain the natural frequencies which are depicted in **Figs. 17** to **20** .

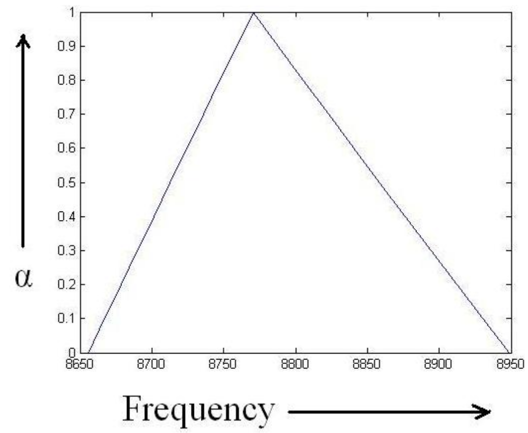



**Fig. 17** First frequency (one element)

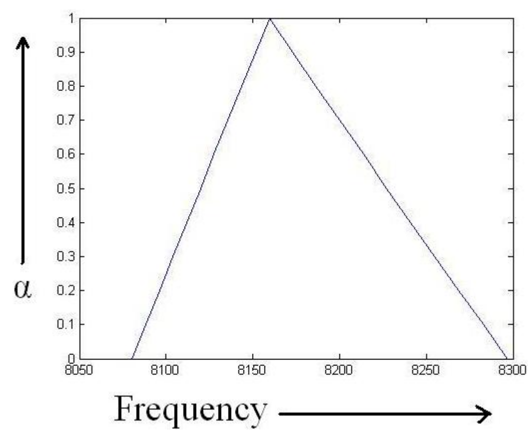



**Fig. 18(a)** First frequency (two element)

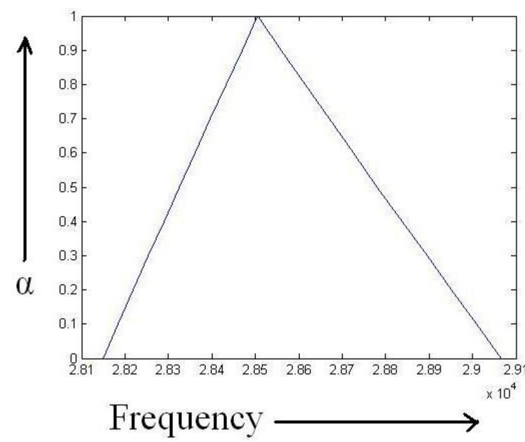
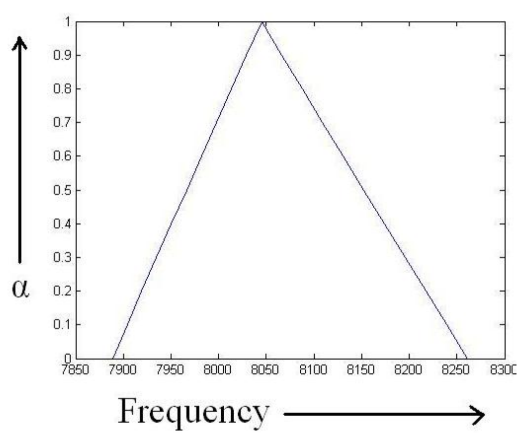



**Fig. 18(b)** Second frequency (two element)



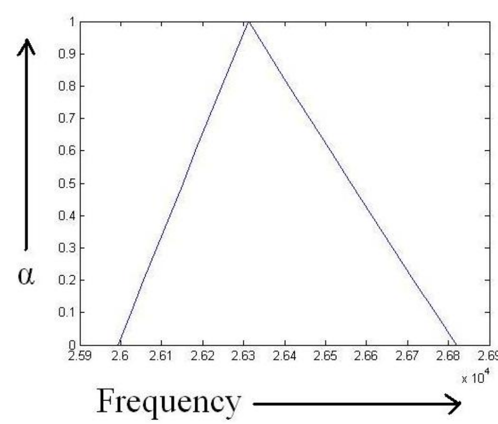



**Fig. 19(a)** First frequency (Three element)

**Fig. 19(b)** Second frequency (Three element)

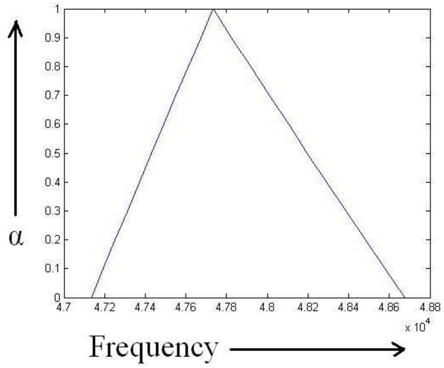
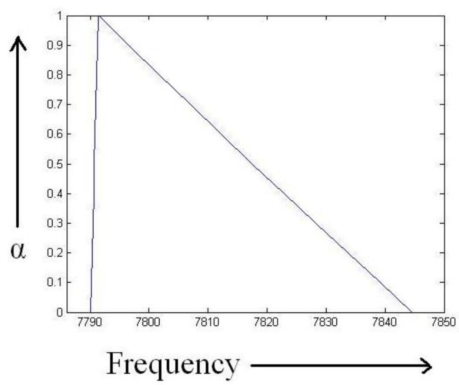



**Fig. 19(c)** Third frequency (Three element)



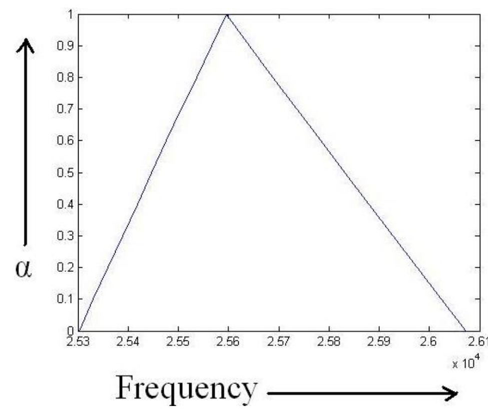
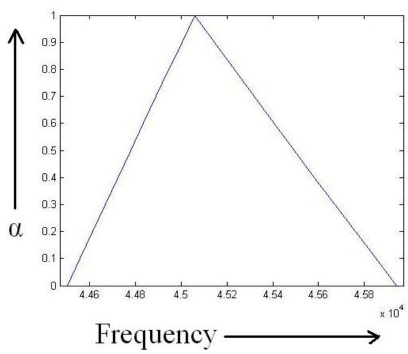




**Fig. 20(a)** First frequency (Four element)

**Fig. 20(b)** Second frequency (Four element)



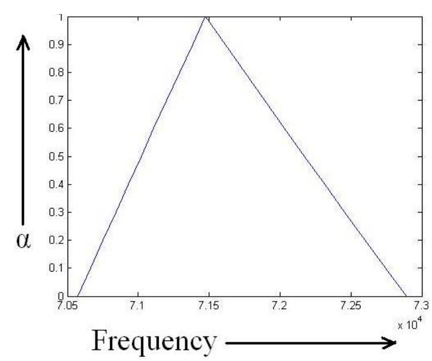




**Fig. 20(c)** Third frequency (Four element)

**Fig. 20(d)** Fourth frequency (Four element)

## 8 Fuzzy Finite element model of Homogenous bar for trapezoidal fuzzy number

In this head fuzzy values (in term of trapezoidal fuzzy number) of the material properties are considered.

### 8.1 Homogenous fixed free bar with fuzzy Young's modulus

A homogenous fixed free bar with fuzzy value of Young's modulus is considered here. If $E=(a_1,b_1,c_1,d_1)$ is a trapezoidal fuzzy number then it can be written in interval form as $[\alpha(b_1-a_1)+a_1, d_1-\alpha(d_1-c_1)]$, where $\alpha\in[0,1]$. The governing equations satisfying the boundary condition are same as in interval case in Eqs. (13) to (16) with $E=[\underline{E},\overline{E}]$ $=[\alpha(b_1-a_1)+a_1, d_1-\alpha(d_1-c_1)]$

Here the values of the parameters are considered here as:
$L=1m, E=(1.998\times10^{11}, 1.999\times10^{11}, 2.001\times10^{11}, 2.002\times10^{11})\ N/m^2, \rho=7800\ kg/m^3$ and $A=30\times10^{-6}\,m^2$. Corresponding natural frequencies are computed for one, two, three and four element which are fuzzy numbers (TrFN). **Figs. 21** to**24** depicts the trapezoidal fuzzy natural frequency plot for different elements.

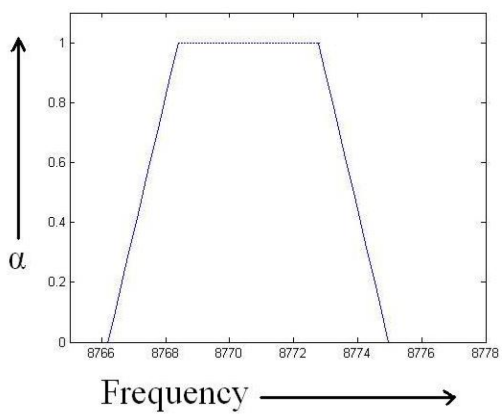



**Fig.21** Natural Frequency (one element)

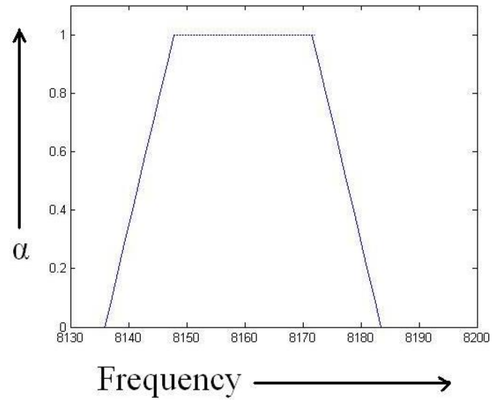



**Fig. 22(a)** First frequency (two element)

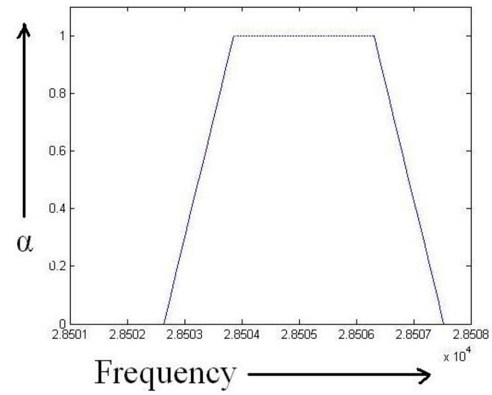



**Fig. 22(b)** Second frequency (two element)

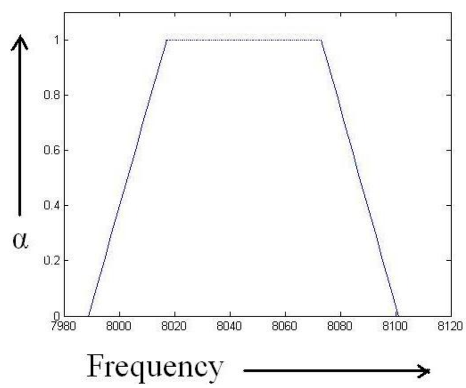



**Fig. 23(a)** First frequency (three element)

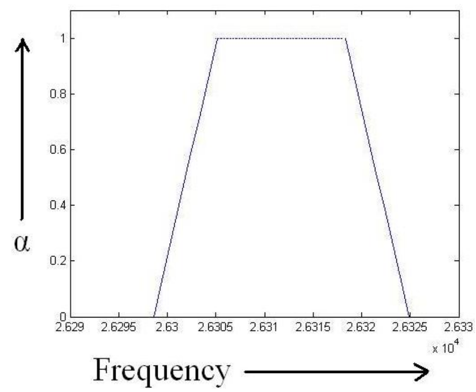



**Fig. 23(b)** Second frequency (three element)

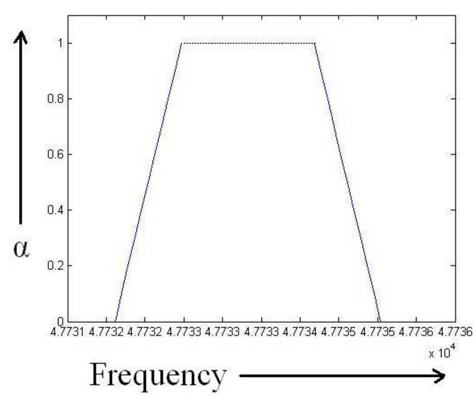



**Fig. 23(c)** Third frequency (three element)

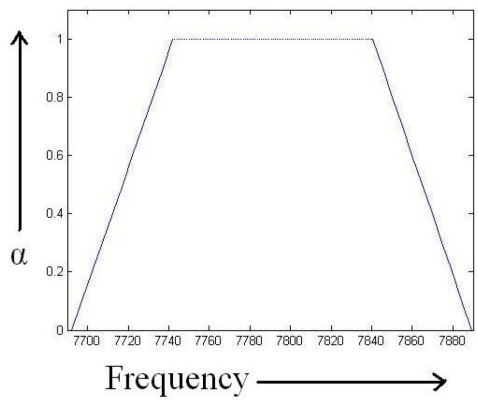



**Fig. 24(a)** First frequency (four element)

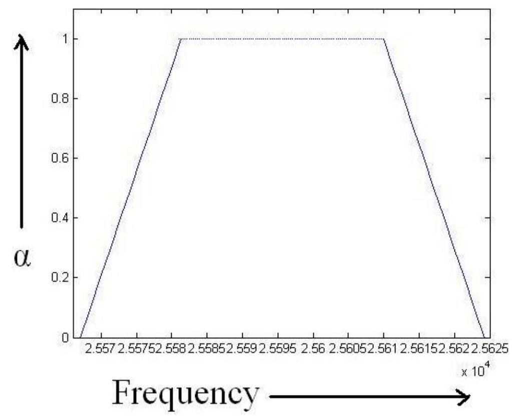



**Fig. 24(b)** Second frequency (four element)

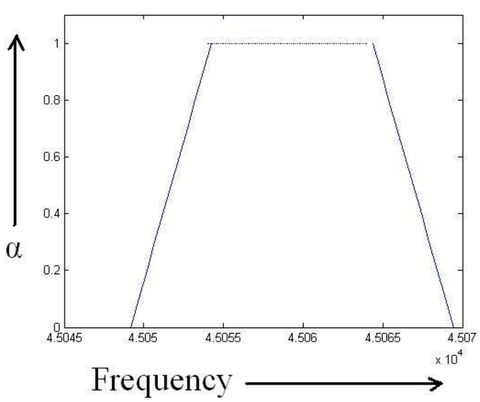



**Fig. 24(c)** Third frequency (four element)

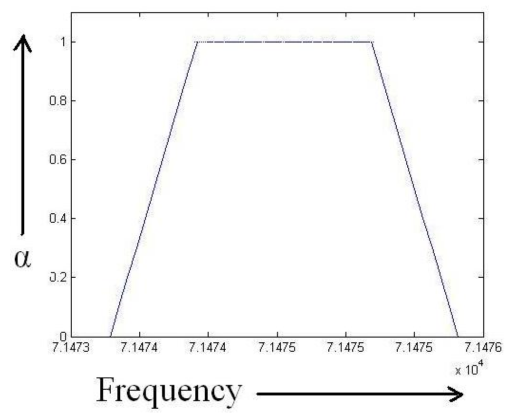



**Fig. 24(d)** Fourth frequency (four element)

## 8.2 Homogenous fixed free bar with fuzzy density

Here the same homogenous bar with fuzzy value of density is considered. Accordingly if $\rho = (a_2, b_2, c_2, d_2)$ is a trapezoidal fuzzy number then its interval form is $[\alpha(b_2 - a_2) + a_2, d_2 - \alpha(d_2 - c_2)]$. The governing equations satisfying the boundary condition are same as in interval case in Eqs. (17) to (20) with $\rho = [\underline{\rho}, \overline{\rho}] = [\alpha(b_2 - a_2) + a_2, d_2 - \alpha(d_2 - c_2)]$.

Values of the parameters are considered in this case as:

$L = 1m$, $E = 2 \times 10^{11}\, N/m^2$, $\rho = (7500, 7700, 7900, 8000)\, kg/m^3$ and $A = 30 \times 10^{-6}\, m^2$.

Corresponding natural frequencies are computed for one, two, three and four element which are fuzzy number (TrFN). The fuzzy natural frequency plots for different elements are depicted in term of plots from **Figs. 25** to **28.**

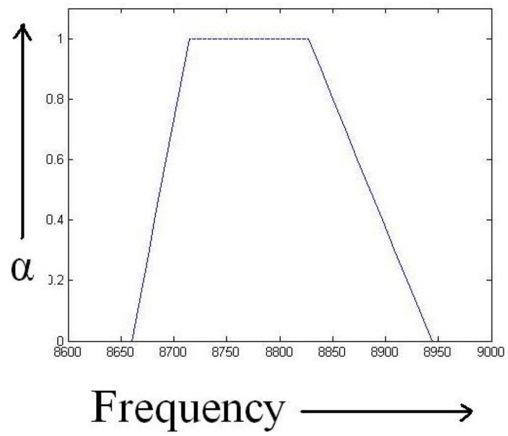



**Fig.25** Natural Frequency (one element)

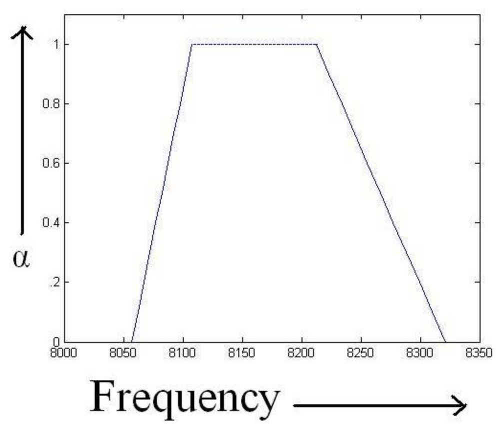



**Fig. 26(a)** First frequency (two element)

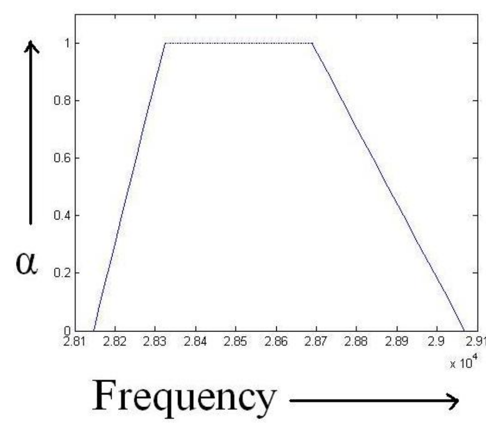



**Fig. 26(b)** Second frequency (two element)

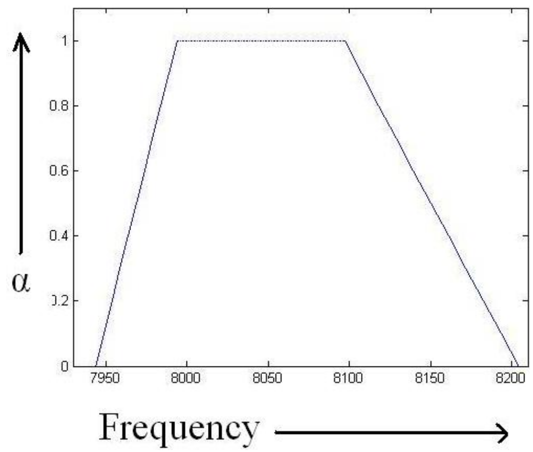



**Fig. 27(a)** First frequency (three element)

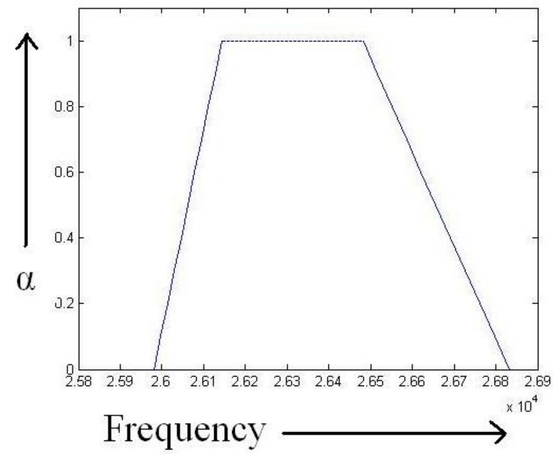



**Fig. 27(b)** Second frequency (three element)

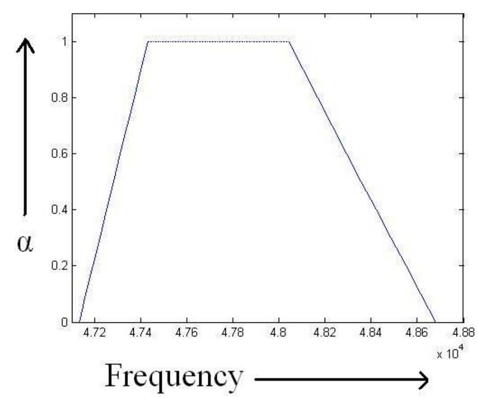



**Fig. 27(c)** Third frequency (three element)

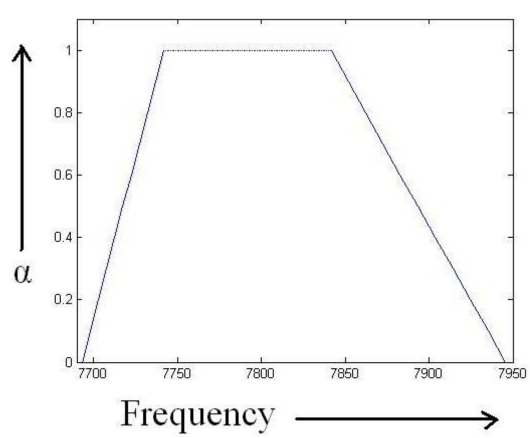



**Fig. 28(a)** First frequency (four element)

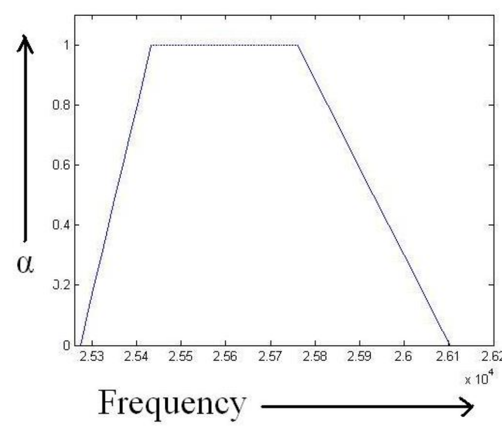



**Fig. 28(b)** Second frequency (four element)

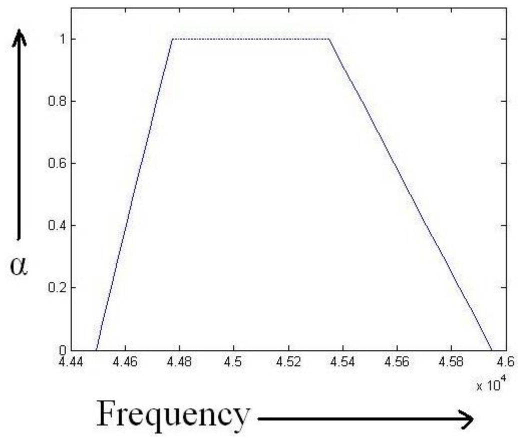



**Fig. 28(c)** Third frequency (four element)

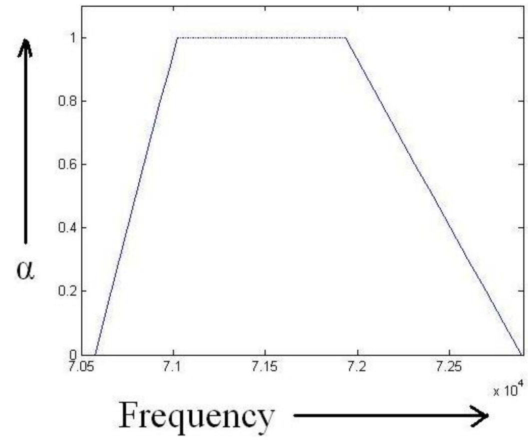



**Fig. 28(d)** Fourth frequency (four element)

### 8.3 Homogenous fixed free bar with fuzzy Young's modulus and fuzzy density

A fixed free bar with fuzzy values of density and Young's modulus is considered. If $E=(a_1,b_1,c_1,d_1)$ and $\rho=(a_2,b_2,c_2,d_2)$ are Trapezoidal Fuzzy Numbers (TrFN) (Ross [14]) then their corresponding interval forms in term of α-cut are $[\alpha(b_1-a_1)+a_1, d_1-\alpha(d_1-c_1)]$ and $[\alpha(b_2-a_2)+a_2, d_2-\alpha(d_2-c_2)]$. The eigenvalue equations satisfying the boundary condition are same as in interval case in Eqs. (21) to (24) with $\underline{E}=\alpha(b_1-a_1)+a_1, \overline{E}=d_1-\alpha(d_1-c_1)$, $\underline{\rho}=\alpha(b_2-a_2)+a_2$ and $\overline{\rho}=d_2-\alpha(d_2-c_2)$.

The values of the parameters are considered as: $\rho=(7500,7700,7900,8000)\,kg/m^3$, $L=1m$, $E=(1.998\times10^{11},1.999\times10^{11},2.001\times10^{11},2.002\times10^{11})\ N/m^2$,and $A=30\times10^{-6}\,m^2$.
Corresponding natural frequencies are computed which are fuzzy number (TrFN) and depicted by **Figs. 29** to **32.**

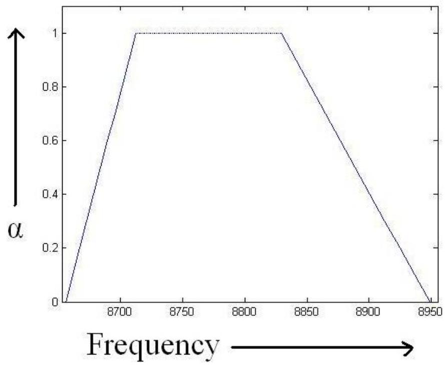



**Fig. 29** First frequency (one element)

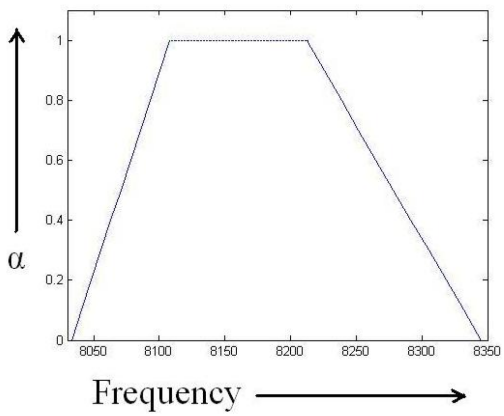



**Fig. 30(a)** First frequency (two element)

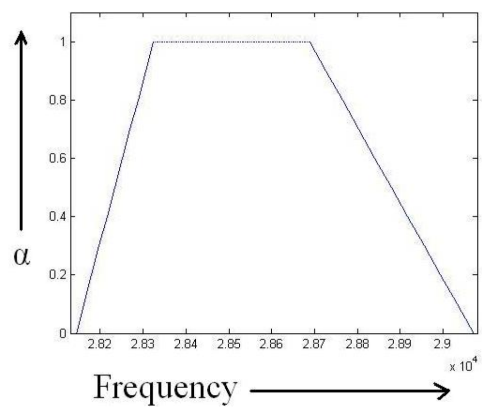



**Fig. 30(b)** Second frequency (two element)

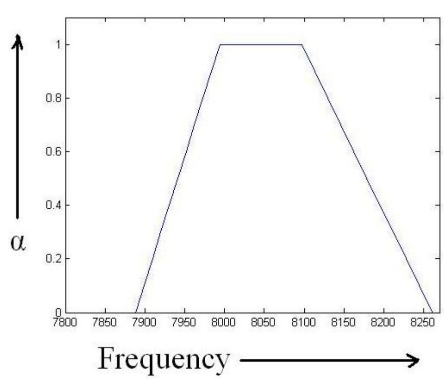



**Fig. 31(a)** First frequency (Three element)

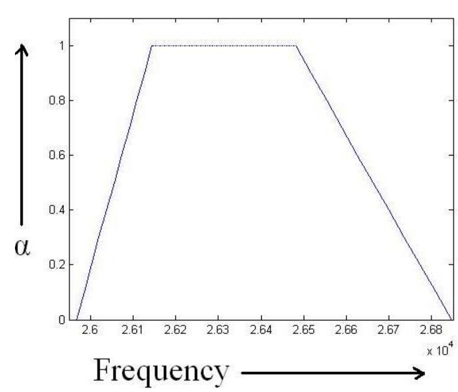



**Fig. 31(b)** Second frequency (Three element)

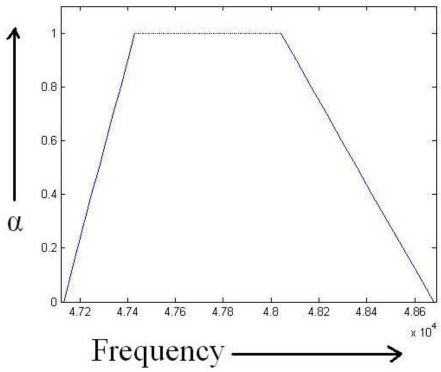



**Fig. 31(c)** Third frequency (Three element)

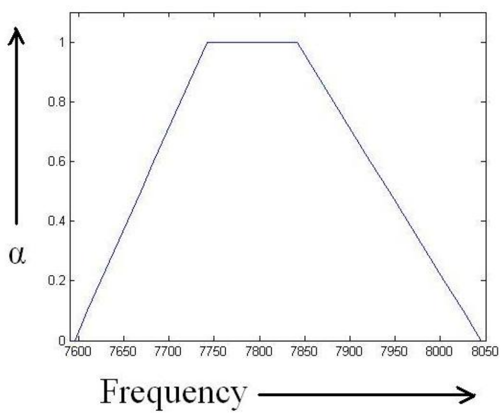



**Fig. 32(a)** First frequency (Four element)

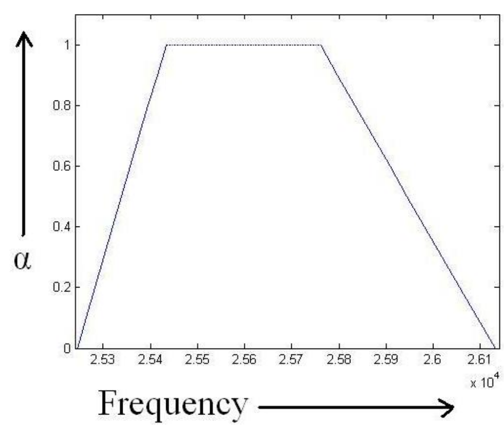



**Fig. 32(b)** Second frequency (Four element)

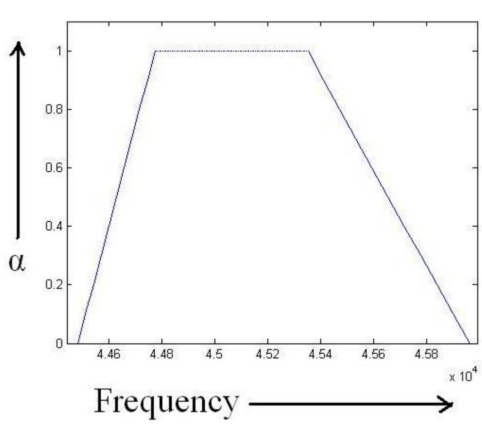



**Fig. 32(c)** Third frequency (Four element)

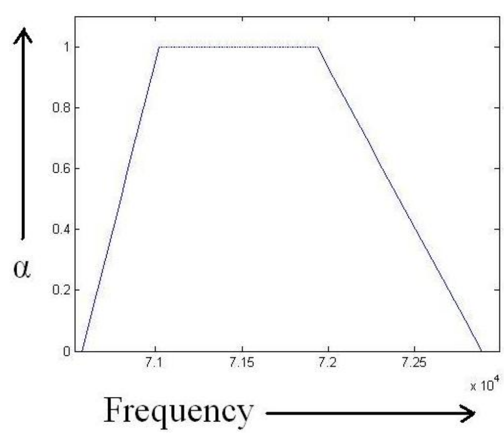



**Fig. 32(d)** Fourth frequency (Four element)

## 9 Finite element model for non-homogeneous bar with crisp material properties

A non-homogenous bar having crisp material properties is considered in this head. The Young's modulus and density varies for different elements along the bar. As such the global mass and stiffness matrices for one, two, three and four element equations are given in Eqs. (25) to (28) respectively.

$$\frac{E_1 A}{L} u_2 = \frac{\rho_1 A L}{3} u_2 \tag{25}$$

$$\frac{2A}{L}\begin{bmatrix} E_1+E_2 & -E_2 \\ -E_2 & E_2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{AL}{12}\begin{bmatrix} 2(\rho_1+\rho_2) & \rho_2 \\ \rho_2 & 2\rho_2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{26}$$

$$\frac{3A}{L}\begin{bmatrix} E_1+E_2 & -E_2 & 0 \\ -E_2 & E_2+E_3 & -E_3 \\ 0 & -E_3 & E_3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \omega^2 \frac{AL}{18}\begin{bmatrix} 2(\rho_1+\rho_2) & \rho_2 & 0 \\ \rho_2 & 2(\rho_2+\rho_3) & \rho_3 \\ 0 & \rho_3 & 2\rho_3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{27}$$

$$\frac{4A}{L}\begin{bmatrix} E_1+E_2 & E_2 & 0 & 0 \\ E_2 & E_2+E_3 & E_3 & 0 \\ 0 & E_3 & E_3+E_4 & E_4 \\ 0 & 0 & E_4 & E_4 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \omega^2 \frac{\rho AL}{24}\begin{bmatrix} 2(\rho_1+\rho_2) & \rho_2 & 0 & 0 \\ \rho_2 & 2(\rho_2+\rho_3) & \rho_3 & 0 \\ 0 & \rho_3 & 2(\rho_3+\rho_4) & \rho_4 \\ 0 & 0 & \rho_4 & 2\rho_4 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \quad \ldots \tag{28}$$

For values $L = 1m, E_1 = 2\times10^{11} N/m^2, E_2 = 3\times10^{11} N/m^2, E_3 = 4\times10^{11} N/m^2$ , $E_4 = 1\times10^{11} N/m^2$ , $\rho_1 = 7800\ kg/m^3$, $\rho_2 = 8200\ kg/m^3, \rho_3 = 7500\ kg/m^3, \rho_4 = 8500\ kg/m^3$ and $A = 30\times10^{-6}\ m^2$, we obtain natural frequencies from Eqs. (25) to (28) and are given in **Table 5**.

**Table 5** Crisp values for natural frequencies for non-homogenous bar

| | | Number of elements | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Modes | 1 | 8770.6 | 8195 | 8636 | 1034.7 |
| | 2 | | 33533.565 | 3158.1 | 2397 |
| | 3 | | | 6436.2 | 3883.6 |
| | 4 | | | | 7305.1 |

## 10 Interval Finite element model for non-homogenous fixed free bar

In this case, the same non-homogenous bar with both density and Young's modulus as interval is considered. Equations for different number of elements are developed. Here, the governing equations satisfying the boundary conditions are given for one, two, three and four elements in Eqs. (29) to (32). Accordingly $\rho$ and $E$ for different elements along the bar are taken as

$E_1=\left[\underline{E}_1,\overline{E}_1\right]$, $E_2=\left[\underline{E}_2,\overline{E}_2\right]$, $E_3=\left[\underline{E}_3,\overline{E}_3\right]$, $E_4=\left[\underline{E}_4,\overline{E}_4\right]$, $\rho_1=[\underline{\rho}_1,\overline{\rho}_1]$, $\rho_2=[\underline{\rho}_2,\overline{\rho}_2]$,

$\rho_3=[\underline{\rho}_3,\overline{\rho}_{31}]$ and $\rho_4=[\underline{\rho}_4,\overline{\rho}_4]$

$$\frac{\underline{E}_1 A}{L}=\underline{\omega}^2\,\frac{\overline{\rho}_1 AL}{3}\ \text{ and }\ \frac{\overline{E}_1 A}{L}=\overline{\omega}^2\,\frac{\underline{\rho}_1 AL}{3} \tag{29}$$

$$\frac{A}{L}\begin{bmatrix}2(\underline{E}_1+\underline{E}_2) & -2\overline{E}_2\\ -2\overline{E}_2 & 2\underline{E}_2\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\underline{\omega}^2\,\frac{AL}{12}\begin{bmatrix}2(\overline{\rho}_1+\overline{\rho}_2) & \overline{\rho}_2\\ \overline{\rho}_2 & 2\overline{\rho}_2\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}\ \text{and}$$

$$\frac{A}{L}\begin{bmatrix}2(\overline{E}_1+\overline{E}_2) & -2\underline{E}_2\\ -2\underline{E}_2 & 2\overline{E}_2\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}=\overline{\omega}^2\,\frac{AL}{12}\begin{bmatrix}2(\underline{\rho}_1+\underline{\rho}_2) & \underline{\rho}_2\\ \underline{\rho}_2 & 2\underline{\rho}_2\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix} \tag{30}$$

$$\frac{3A}{L}\begin{bmatrix}\underline{E}_1+\underline{E}_2 & -\overline{E}_2 & 0\\ -\overline{E}_2 & \underline{E}_2+\underline{E}_3 & -\overline{E}_3\\ 0 & -\overline{E}_3 & \underline{E}_3\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\end{Bmatrix}=\underline{\omega}^2\,\frac{AL}{18}\begin{bmatrix}2(\overline{\rho}_1+\overline{\rho}_2) & \overline{\rho}_2 & 0\\ \overline{\rho}_2 & 2(\overline{\rho}_2+\overline{\rho}_3) & \overline{\rho}_3\\ 0 & \overline{\rho}_3 & 2\overline{\rho}_3\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\end{Bmatrix}\ \text{and}$$

$$\frac{3A}{L}\begin{bmatrix}\overline{E}_1+\overline{E}_2 & -\underline{E}_2 & 0\\ -\underline{E}_2 & \overline{E}_2+\overline{E}_3 & -\underline{E}_3\\ 0 & -\underline{E}_3 & \overline{E}_3\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\end{Bmatrix}=\overline{\omega}^2\,\frac{\rho AL}{18}\begin{bmatrix}2(\underline{\rho}_1+\underline{\rho}_2) & \underline{\rho}_2 & 0\\ \underline{\rho}_2 & 2(\underline{\rho}_2+\underline{\rho}_3) & \underline{\rho}_3\\ 0 & \underline{\rho}_3 & 2\underline{\rho}_3\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\end{Bmatrix} \tag{31}$$

$$\frac{4A}{L}\begin{bmatrix}\underline{E}_1+\underline{E}_2 & \overline{E}_2 & 0 & 0\\ \overline{E}_2 & \underline{E}_2+\underline{E}_3 & \overline{E}_3 & 0\\ 0 & \overline{E}_3 & \underline{E}_3+\underline{E}_4 & \overline{E}_4\\ 0 & 0 & \overline{E}_4 & \underline{E}_4\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\\u_5\end{Bmatrix}=\underline{\omega}^2\,\frac{\rho AL}{24}\begin{bmatrix}2(\overline{\rho}_1+\overline{\rho}_2) & \overline{\rho}_2 & 0 & 0\\ \overline{\rho}_2 & 2(\overline{\rho}_2+\overline{\rho}_3) & \overline{\rho}_3 & 0\\ 0 & \overline{\rho}_3 & 2(\overline{\rho}_3+\overline{\rho}_4) & \overline{\rho}_4\\ 0 & 0 & \overline{\rho}_4 & 2\overline{\rho}_4\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\\u_5\end{Bmatrix}$$

$$\frac{4A}{L}\begin{bmatrix} \overline{E}_1+\overline{E}_2 & \underline{E}_2 & 0 & 0 \\ \underline{E}_2 & \overline{E}_2+\overline{E}_3 & \underline{E}_3 & 0 \\ 0 & \underline{E}_3 & \overline{E}_3+\overline{E}_4 & \underline{E}_4 \\ 0 & 0 & \underline{E}_4 & \overline{E}_4 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \overline{\omega}^2 \frac{\rho A L}{24}\begin{bmatrix} 2(\underline{\rho}_1+\underline{\rho}_2) & \underline{\rho}_2 & 0 & 0 \\ \underline{\rho}_2 & 2(\underline{\rho}_2+\underline{\rho}_3) & \underline{\rho}_3 & 0 \\ 0 & \underline{\rho}_3 & 2(\underline{\rho}_3+\underline{\rho}_4) & \underline{\rho}_4 \\ 0 & 0 & \underline{\rho}_4 & \underline{\rho}_4 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \quad \ldots (32).$$

For values $E_1=[1.998\times10^{11},2.002\times10^{11}]N/m^2$, $E_2=[2.998\times10^{11},3.002\times10^{11}]N/m^2$, $E_3=[3.998\times10^{11},4.002\times10^{11}]$ $N/m^2$, $E_4=[0.998\times10^{11},1.002\times10^{11}]N/m^2$, $L=1m$ $\rho_1=[7500,8000]\,kg/m^3$, $\rho_2=[8000,8500]\,kg/m^3$, $\rho_3=[7200,7700]\,kg/m^3$, $\rho_4=[8200,8700]\,kg/m^3$ and $\mathrm{A}=30\times10^{-6}\,\mathrm{m}^2$ natural frequencies are computed for various elements and are incorporated in **Table 6.**

**Table 6** Interval values of frequencies for $\rho, E$ as intervals

| | Number of elements | | | | |
|---|---|---|---|---|---|
| Modes | | 1 | 2 | 3 | 4 |
| | 1 | (8655.9, 8948.7 ) | ( 8036, 8290 ) | ( 8456, 8818 ) | (1009.6 , 1061.8) |
| | 2 | | ( 3299.9, 3390.7 ) | (3111.7, 3209.2 ) | ( 2366.2, 2441.4 ) |
| | 3 | | | ( 6342, 6537 ) | (3827 , 3945 ) |
| | 4 | | | | ( 7199.8, 7415.9 ) |

## 11 Fuzzy Finite element model for non-homogenous bar

A non-homogenous fixed free bar with fuzzy values (TFN) of density and Young's modulus is considered here. If $E_1=(a_1,b_1,c_1)$, $E_2=(a_2,b_2,c_2)$, $E_3=(a_3,b_3,c_3)$, $E_4=(a_4,b_4,c_4)$, $\rho_1=(d_1,e_1,f_1)$, $\rho_2=(d_2,e_2,f_2)$, $\rho_3=(d_3,e_3,f_3)$ and $\rho_4=(d_4,e_4,f_4)$ are triangular fuzzy numbers (Ross [11]) for the mentioned materials in the different elements then their corresponding interval forms in term of α-cut are respectively given as

$$[\alpha(b_1-a_1)+a_1, c_1-\alpha(c_1-b_1)],\ [\alpha(b_2-a_2)+a_2, c_2-\alpha(c_2-b_2)],$$
$$[\alpha(b_3-a_3)+a_3, c_3-\alpha(c_3-b_3)],\ [\alpha(b_4-a_4)+a_4, c_4-\alpha(c_4-b_4)],$$
$$[\alpha(e_1-d_1)+d_1, f_1-\alpha(f_1-e_1)],\ [\alpha(e_2-d_2)+d_2, f_2-\alpha(f_2-e_2)],$$
$$[\alpha(e_3-d_3)+d_3, f_3-\alpha(f_3-e_3)] \text{ and } [\alpha(e_4-d_4)+d_4, f_4-\alpha(f_4-e_4)]$$

The eigenvalue equations satisfying the boundary conditions are same as in interval case in equations (26), (27), (28) and (29) with $\underline{E}_1=\alpha(b_1-c_1)+a_1$, $\overline{E}_1=c_1-\alpha(c_1-b_1)$, $\underline{E}_2=\alpha(b_2-c_2)+a_2$, $\overline{E}_2=c_2-\alpha(c_2-b_2)$, $\underline{\rho}_1=\alpha(e_1-d_1)+d_1$, $\overline{\rho}_1=f_1-\alpha(f_1-e_1)$, $\underline{\rho}_2=\alpha(e_2-d_2)+d_2$ and $\overline{\rho}_2=f_2-\alpha(f_2-e_2)$

$$\underline{E}_1=\alpha(b_1-c_1)+a_1,\ \overline{E}_1=c_1-\alpha(c_1-b_1),\ \underline{E}_2=\alpha(b_2-c_2)+a_2, \overline{E}_2=c_2-\alpha(c_2-b_2)\ ,$$
$$\underline{E}_3=\alpha(b_3-c_3)+a_3\ ,\ \overline{E}_3=c_3-\alpha(c_3-b_3)\ \underline{E}_4=\alpha(b_4-c_4)+a_2\ \overline{E}_4=c_4-\alpha(c_4-b_4)$$
$$\underline{\rho}_1=\alpha(e_1-d_1)+d_1, \overline{\rho}_1=f_1-\alpha(f_1-e_1),\ \underline{\rho}_2=\alpha(e_2-d_2)+d_2,\ \underline{\rho}_2=\alpha(e_2-d_2)+d_2,$$
$$\underline{\rho}_3=\alpha(e_3-d_3)+d_3,\ \overline{\rho}_3=f_3-\alpha(f_3-e_3),\ \overline{\rho}_4=f_4-\alpha(f_4-e_4) \text{ and } \overline{\rho}_4=f_4-\alpha(f_4-e_4)$$

Here the values of the parameters are considered as:

$E_1=(1.998\times10^{11}, 2\times10^{11}, 2.002\times10^{11})\ N/m^2$, $E_2=(2.998\times10^{11}, 3\times10^{11}, 3.002\times10^{11})\ N/m^2$, $E_3=(3.998\times10^{11}, 4\times10^{11}, 4.002\times10^{11})\ N/m^2$, $E_4=(0.998\times10^{11}, 1\times10^{11}, 1.002\times10^{11})\ N/m^2$, $\rho_1=(7500, 7800, 8000)\ \mathrm{kg/m^3}$, $\rho_2=(8100, 8200, 8500)\ kg/m^3$, $\rho_3=(7200, 7500, 7700)\ kg/m^3$, $\rho_4=(8200, 8500, 8700)\ kg/m^3$, $A=30\times10^{-6}\ m^2$ and $L=1m$. Corresponding natural frequencies are computed and are shown in **Figs. 29** to **32**.

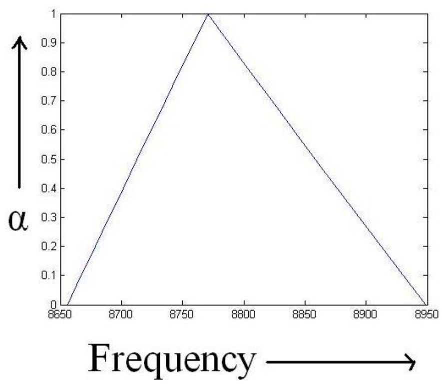



**Fig. 29** First frequency (one element)

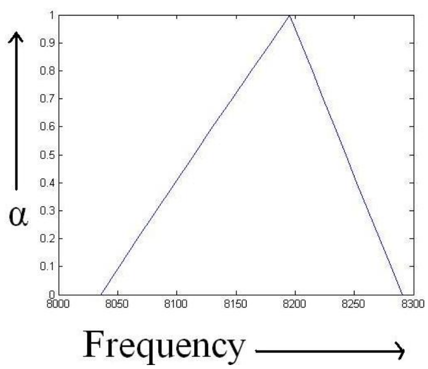



**Fig. 30(a)** First frequency (two element)

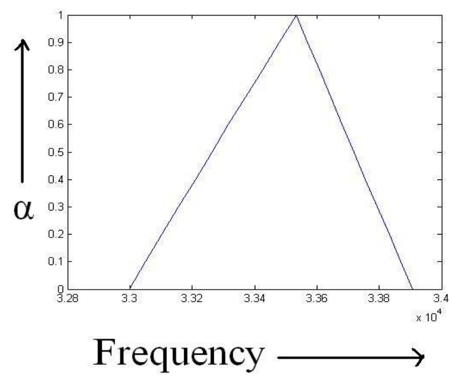



**Fig. 30(b)** Second frequency (two element)

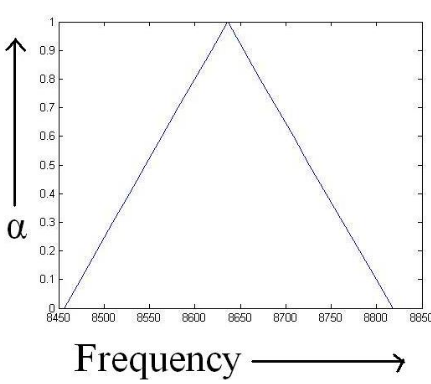



**Fig. 31(a)** First frequency (three element)

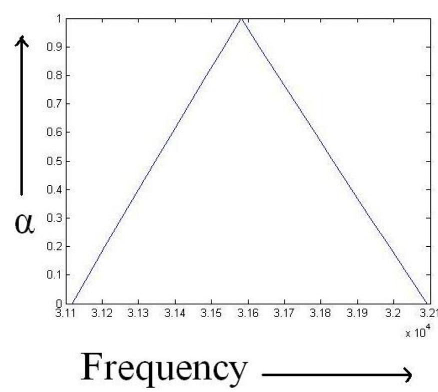



**Fig. 31(b)** Second frequency (three element)

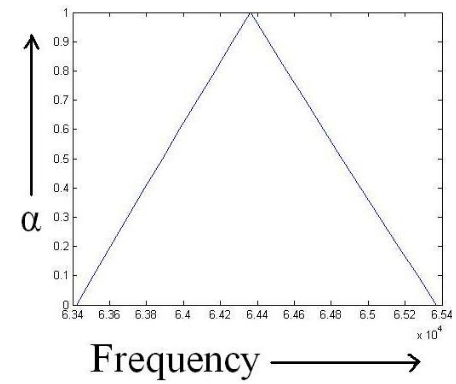



**Fig. 31(c)** Third frequency (three element)

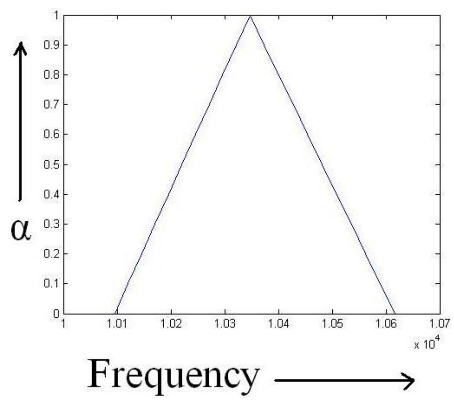



**Fig. 32(a)** First frequency (four element)

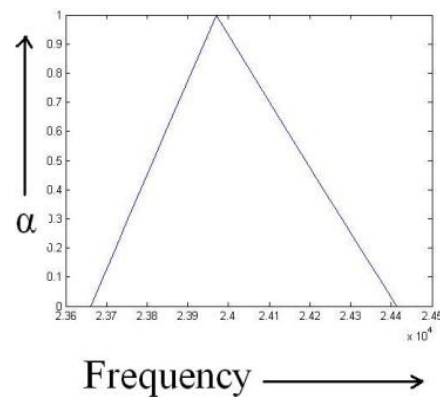



**Fig. 32(b)** Second frequency (four element)

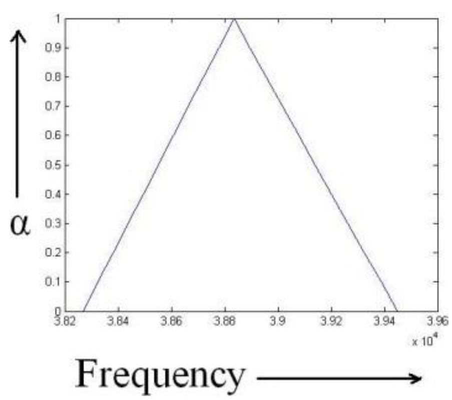



**Fig. 32(c)** Third frequency (four element)

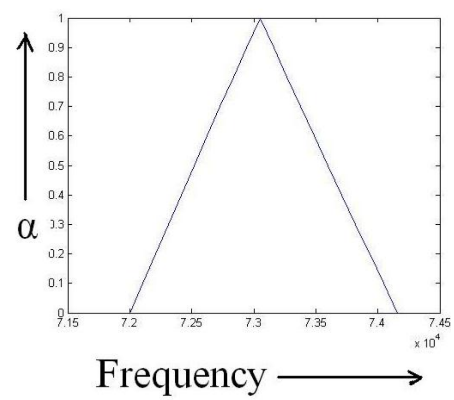



**Fig. 32(d)** Fourth frequency (four element)

## 12 Special cases

One may note the point related to the eigenvalue equations that result for crisp case can be obtained simply by putting $\alpha = 1$in fuzzy case (TFN). Moreover by substituting $\alpha = 0$in fuzzy case we can obtain the results for the interval case.

Also one can note that for $\alpha = 1$in fuzzy case (TrFN) given by $A = (a, b, c, d)$ if $b = c$ then it gets reduced to triangular fuzzy number.

## 13 Discussions

It may be seen from the above numerical results that the natural frequencies gradually decrease with increase in number of elements as it should be. In crisp values of natural frequency for homogenous and non-homogenous bar, the first natural frequency got reduced to 8006.248 from 8770.32. Similar trend of reduction may also be seen for interval and fuzzy cases. Moreover, in Table 3 the interval width for natural frequencies also reduces with increase in elements (first natural frequency reduces to (7693, 7946) from (8660.3, 8944.3). This is true for only in the density case. However in case of Young's modulus (as interval) it is increasing. The case of Young's modulus and density both as interval at a time the width again increase as we increase the number of elements. It is interesting to note also that the addition of the computed frequency widths for the cases of homogeneous bar viz. interval/fuzzy $E$ and $\rho$ (such as Tables 2 and 3) gives the interval width of natural frequencies in Table 4. Moreover using Eqs. (11) and (12), the computed natural frequencies in Table 4.2 has a much lesser interval width than in Table 4.1 by using only the Eq.(11).

## 14 Conclusions

The investigation presents here the Fuzzy FEM in the vibration of a fixed free bar. The related generalized eigenvalue problem with respect to the interval and fuzzy components are solved to obtain the natural frequencies depending upon the number of elements taken in the discretization. Two types of fuzzy number viz. triangular and trapezoidal fuzzy numbers are considered for the analysis. Two methods are given to obtain interval or fuzzy eigenvalues. It may be worth mentioning that the second method gives better result in term of interval width. The investigation presented here may find in real application where the material properties may not be obtained in term of crisp values but a vague value in term of either interval or fuzzy is known.

## 15 Future directions

The investigation gives a new idea of the Fuzzy FEM through eigenvalue computation and this can very well be used in future research for better results for other eigenvalue problems obtained in different applications. The idea may easily be extended to other structural problems with various complicating effects. Although this require more complex form of interval and fuzzy computation to handle the corresponding problem.

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## List of Publications/Communicated

1. Mahato N. R, Behera D and Chakraverty S, Fuzzy Finite Element Method for Vibration of a Bar, $38^{th}$ annual conference of Orissa Mathematical Society, ITER Bhubaneswar, Jan 8-9,2011.( **Best Paper Presentation Award** )
2. Mahato N. R and Chakraverty S Fuzzy Finite Element Method for Vibration Analysis of an imprecisely defined Bar, Meccanica, Springer , 2011 (Communicated)